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Investigation of Goldbach's Conjecture with the Congruence Primality Theorems and Complementary Congruence

Abstract

This article provides novel insights on Goldbach's conjecture; this work is primarily based on two primality theorems of congruence and of compcongruence. Study results in demonstration of the Goldbach conjecture.The approach taken also opens up new areas of possible research in the field of Number Theory.

1 The Proof of the Goldbach Conjecture

Goldbach's conjecture assumes that for every even number $2N_0$ there exist one or more numbers n ∈ N such that N_0 - n and N_0 + n are two prime numbers whose sum is obviously equal to $2N_0$.

Given an N_0 ∈ N we denote by the letter Ꞡ every number n ∈ N such that N_0 - n and N_0 + n are two prime numbers.

1.1 The Ꞡ Number Theorem of N_0

Definition 1.1 *∀ N , n_{00} ∈ N and n_0 even if N_0 is odd or vice versa, with $N_0 \geq 9$, $0 \leq n_0 \leq N_0 - p_{,max}$ with $_{pmax}$ being a prime number higher than $\mathbb{P}\left(\sqrt{2N_0}\right)$, where $\mathbb{P}\left(\sqrt{2N_0}\right)$ è l' set of odd prime numbers $\leq \sqrt{(2N_0)}$, a necessary and sufficient condition for $N_0 - n_0$ and $N_0 + n_0$ to be two prime numbers is that n_0 is an incongruous and incompcongruous number of N_0 (1.2.3 and 1.4.3 [c]).*

Dim. From the Corollaries (1.2.3 [c]) and (1.4.3 [c]), placing the most restrictive conditions between the two, derive the necessary and sufficient conditions of the Theorem. Just as from Observations (1.2.5) and (1.4.4) it follows that there certainly exists at least one n_{01} incongruous and at least one n_{02} incompcongruous of N_0 but we cannot derive from them that esiste anche un $n_0 = n_{01} = n_{02}$.
To prove Goldbach's conjecture, on the other hand, it must be established that for every $N_0 \geq 9$ there exists at least un $n_0 = n_{01} = n_{02}$ i.e. an incongruous and incompcongruous number Ꞡ of N_0 .

Apart from the special case of a prime N_0 and thus the certain existence of a Ꞡ = 0, we must therefore prove that for every N_0 there always exists an incongruous and incompcongruous Ꞡ of N_0 and thus that there always exist two prime numbers equidistant from N_0 :

$p_1 = N_0$ - Ꞡ

$p_2 = N_0$ + Ꞡ

and whose sum is evidently equal to $2N_0$.

To this end, we resort to the study of the density of numbers Ꞡ.

1.2 The density of numbers Ꞡ

Let us say right away that each Ꞡ must have the following characteristics:

- its class of modulus 2 must be equal to zero if N_0 is odd, to 1 if N_0 is even;
- its successive first module classes (3, 5, 7, etc.) less than or equal to the $(\sqrt{2N_0})$ must not be equal to the two classes corresponding to the remainder (for non-congruence) and its complement (for non-compcongruence) of N_0 for the same modules (*e.g. if* $N_0 = 43$ *and* Ꞡ*=30 we have that* $\mathbb{P}(\sqrt{N_0})$ *= {3,5}; [43]*$_{mod3}$ *= 1 with a complement equal to 2, [43]*$_{mod5}$ *= 3 with a complement equal to 2; [30]* $_{mod3}$ *=[0] and [30]* $_{mod5}$ *=[0]; therefore* Ꞡ *is prisubordinate and prisopordinate to* N_0 *and therefore 73 (43+30) and 13 (43-30) constitute a pair of primes whose sum is equal to* $2N_0$).

Having said this, let us see how to calculate the number of Ꞡ less than an $N_0 \geq 121$ (a condition arising as we know (1.7.2 [c]) from the requirement that $2N_0$ belongs to the interval [0, $\sqrt{(\boldsymbol{2N_0})}$ #]).
Having then selected any $N_0 \geq 121$, we call p_{max} the highest prime number less than or equal to the $\sqrt{(\boldsymbol{2N_0})}$. We then consider the table-interval of natural numbers [0, $\sqrt{(\boldsymbol{2N_0})}$ #] = [0, p_{max} #], where p_{max} is the highest prime number less than $\sqrt{(\boldsymbol{2N_0})}$, and p_{max} # corresponds to the product 2*3*5*.......* p_{max} , a product that corresponds to the last number in the relevant number-class table p_{max} (1.5.1 [c]) of bi-univocal correspondence between the numbers in the range and their respective combinations of congruence classes.

Let us now eliminate from this table [0, p_{max} #] each of the rows that has a congruence class mod 2 equal to 0 or 1 depending on whether N_0 is even or odd, and/or congruence classes of the following modules (3, 5, ... , p_{max}) equal to one of the two classes corresponding to the remainder and complement of N_0 for the same modules.
The M-numbers in the table, which were not eliminated through the previous sieve, can then only be:

a) those which in the number-class table p_{max} have in their corresponding combination of congruence classes only one of the two possible congruence classes modulo 2
b) those which in the number-class table p_{max} for each odd p_i belonging to the set $\mathbb{P}(\sqrt{(\boldsymbol{2N_0})})$ and NOT FACTOR of N_0 have in their corresponding combination of congruence classes one of the p_i -2 possible congruence classes of the modules 3, 5, ... , p_{max} that is, with the exclusion of the two classes corresponding to the remainder and the complement of N_0 for the same modules p_i (*if e.g. (N_0) mod7 = 3 with complement = 4, (M) mod7 must be equal to one of the 5 (7-2) other possible congruence classes: 0,1,2,5,6*)
c) those which in the number-class table p_{max} for every odd p_i belonging to the set $\mathbb{P}(\sqrt{(\boldsymbol{2N_0})})$ and FACTOR of N_0 have in their corresponding combination of congruence classes one of the p_i -1 possible congruence classes other than [0] that constitutes both the remainder and the complement of N_0 for the same module-factors.

The numbers N_0 with factors other than p_i odd belonging to the set $\mathbb{P}(\sqrt{(\boldsymbol{2N_0})})$and which therefore fall under category (b) of the previous classification, are the prime numbers outside the set $\mathbb{P}(\sqrt{(\boldsymbol{2N_0})})$ or a multiple of them with coefficient 2^n or a simple power of 2. In particular, let us consider only the prime numbers that we will call N_{0pm} indicating by $\mathbb{P}$ their set.

For the numbers N_{0pm} then the rows (class combinations) of the table [0, p_{max} #] that have not been deleted will, according to the combinatorial calculation, be:

(1.2.1) $\prod_{p=3}^{p_{max}}(p-2)$

Therefore, (1.2.1) gives us the quantity of the numbers M of the table that are incongruent and incompcongruent with N_{0pm} for the p_i belonging to the set $\mathbb{P}\left(\sqrt{(2N_{0pm})}\right)$ while nothing can be said about their possible (non)congruence and/or (non)compcongruence with N_{0pm} with respect to the other modules p_j greater than p_{max} and belonging to the set $\mathbb{P}(\sqrt{(pmax\#)})$.

But this is enough for us to state that according to the Ɠ Number Theorem (1.1.1) we can say that all numbers **M less than N_{0pm} ($M_{Ɠ}$**) are incongruous and incompcongruous than N_{0pm} and are therefore numbers Ɠ.

Remark 1.2.2 *By the corollary (1.2.3 [c]) and remark (1.2.4 [c]) we also know, however, that such numbers $M_{Ɠ}$, (incongruous and incompcongruous of N_{0pm}) do not include the possible n_0 for which (N_{0pm} - n_0) is equal to a p_i belonging to the set $\mathbb{P}\left(\sqrt{(2N_{0pm})}\right)$. Consequently, all numbers Ɠ smaller than N_{0pm} are always greater than/equal to the numbers $M_{Ɠ}$.*

The average density, which we denote by $Dncncomp_{[0,\ \sqrt{2N_{0pm}}\#]}$, of the numbers M existing in the interval $[0, \sqrt{(2N_{0pm})}\#]$ **not congruent with N_{0pm} for only p-modules$_i$** belonging to the set $\mathbb{P}\left(\sqrt{(2N_{0pm})}\right)$, knowing that $\sqrt{(2N_{0pm})}\# = 2*3*.......*\ p_{max}$, it can be written:

(1.2.3) $$Dncncomp_{[0,\ \sqrt{2N_{0pm}}\#]} = \frac{\prod_{p=3}^{pmax}(p-2)}{\prod_{p=2}^{pmax} p} = \frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-2)}{p}$$

At the density $Dncncomp_{[0,\ \sqrt{2N_{0pm}}\#]}$ of incongruous and incompcongruous numbers with N_{0pm} **for only p-modules$_i$** belonging to the set $\mathbb{P}\left(\sqrt{(2N_{0pm})}\right)$ corresponds to a density $Dncncomp_{[0,\ N_{0pm}]}$ of the incongruous and incompcongruous numbers smaller than N_{0pm} and that is of the numbers Ɠ ≤ N_{0pm} and since N_{0pm} is prime, we can state that 0 is definitely one of these numbers Ɠ.

One can therefore write:

(1.2.4) $$Dncncomp_{[0,\ N_{0pm}]} \geq \frac{1}{N_{0pm}}$$

and again, multiplying both members of (1.2.4) by N_{0pm} , the number $M_{Ɠ(N_{0pm})}$ of the numbers Ɠ less than/equal to N_{0pm} :

(1.2.5) $$M_{Ɠ(N_{0pm})} = Dncncomp_{[0,\ N_{0pm}]} * N_{0pm} \geq 1$$

For N_0 other than N_{0pm} the $Dncncomp_{[0,\ \sqrt{2N_0}\#]}$ (1.2.3) is modified in the expression:

(1.2.6) $$Dncncomp_{[0,\ \sqrt{2N_0}\#]} = \frac{1}{2} * \prod_{3\leq p_l \leq p_{max}}\frac{(p_l-2)}{p_l} * \prod_{3\leq p_j \leq p_{max}}\frac{(p_j-1)}{p_j}$$

in which the first p_j belonging to $\mathbb{P}(\sqrt{(2N_0)})$ appear distinct in those p_j equal to the factors of N_0 and those p_l which are not (see section 1.2 (b) and (c)). But (1.2.6) can also be written like this:

(1.2.7) $$Dncncomp_{[0,\ \sqrt{2N_0}\#]} = \frac{1}{2} * \prod_{3\leq p_i \leq p_{max}}\frac{(p_i-2)}{p_i} * \prod_{3\leq p_j \leq p_{max}}\frac{(p_j-1)}{(p_j-2)}$$

Knowing that the value of $p_{\max}$ of (1.2.3) and (1.2.7) remains the same for each interval $[0, N_0\ \#]$ with N_0 such that it results $p_{\max} < \sqrt{2N_0} < p_{\max\,succ}$ where $p_{\max}$ is the highest prime less than

$\sqrt{2N_{0pm}}$ e $p_{\max succ}$ the first immediately following $p_{\max,}$, A comparison between (1.2.7), where $Dncncomp_{[0,\ \sqrt{2N_0}\#]}$ is relative to any N_0 other than N_{0pm} , and (1.2.3) relative to the first N_{0pm} results:

(1.2.8) $Dncncomp_{[0,\ \sqrt{2N_0}\#]} = Dncncomp_{[0,\ \sqrt{2N_{0pm}}\#]} * \prod_{3\le p_j\le p_{max}} \frac{(p_j-1)}{(p_j-2)}$

where both densities refer to the same interval [0, p_{max} #] with p # = $_{max}$ $\sqrt{2N_0}$ # = $\sqrt{2N_{0pm}}$ # but refer respectively to the integers of the interval incongruous and incompcongruous with two different numbers: N_0 and N_{0pm}

Since the term $\prod_{3\le p_j\le p_{max}} \frac{(p_j-1)}{(p_j-2)} > 1$ from (1.2.8) it can be deduced:

(1.2.9) $Dncncomp_{[0,\ \sqrt{2N_0}\#]} > Dncncomp_{[0,\ \sqrt{2N_{0pm}}\#]}$

On the basis of (1.2.9), we can also assume that the density $Dncncomp_{[0,\ N_0]}$ of the incongruous and incompcongruous numbers smaller than N_0 , i.e. of the numbers Ǥ ≤ N_0 in the interval [0, N_0] is greater than that $Dncncomp_{[0,\ N_{0pm}]}$ of the incongruous and incompcongruous numbers less than N_{0pm} in the interval [0, N_{0pm}]. Consequently, it is possible to write:

(1.2.10) $Dncncomp_{[0,\ N_0]} > Dncncomp_{[0,\ N_{0pm}]}$

and according to (1.2.4):

(1.2.11) $Dncncomp_{[0,\ N_0]} \ge \frac{1}{N_{0pm}}$

and again, multiplying both members of (1.2.11) by N_0 , the number $M_{Ǥ(N0)}$ of the numbers Ǥ less than/equal to N_0 , we obtain :

(1.2.12) $M_{Ǥ(N0)} = Dncncomp_{[0,\ N_0]} * N_0 \ge 1$

It therefore follows from (1.2.12) that even for all numbers $N_0 \neq N_{0pm}$ the numbers Ǥ are always greater than or equal to 1 and thus there will always be at least one pair of primes (N_0 - Ǥ and N_0 + Ǥ) whose sum is equal to 2*N_0 as predicted by Goldbach's conjecture.

For N_0 less than 121, Goldbach's conjecture is easily verifiable.

BIBLIOGRAPHY

[a] Alessandro Zaccagnini - Introduction to Analytical Number Theory:
http://people.dmi.unipr.it/alessandro.zaccagnini/psfiles/lezioni/tdn2005.pdf

[b] Francesco Fumagalli - Appunti di Teoria elementare dei numeri:
Theory_of_Numbers.pdf (unifi.it)

[c] Aldo Pappalepore - Congruence, Primality and Density:
https://www.aldopappalepore.it/_downloads/99fd430c6fe12b90fb801cd67dc1d70f