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Investigation of Lemoine and Levy conjectures with the primality theorems of Congruence and of Complementary Congruence

Abstract

In the article, a study on the Lemoine and Levy conjecture is developed that is very similar to the study on the Goldbach conjecture and how it is based on the two primality theorems of congruence and compcongruence. The study arrives at the proof of the Lemoine and Levy conjecture and, similar to the study on the Goldbach conjecture, opens up new areas of possible research in the field of Number Theory.

1 Introduction

The conjecture states that: *"Every odd number greater than 5 can be written as p+2q, where p and q are odd prime numbers that are not necessarily distinct."*

The analogy between Lemoine and Levy's conjecture and Goldbach's conjecture prompts us to follow a similar path of analysis and deepening of this conjecture, a path that led us to intertwine the two demonstrations by extending the two primality theorems of congruence and compcongruence to numbers equal to twice the prime numbers.

The Primality Theorem of Congruence (1.2.1 [c]), and in particular its corollary (1.2.3 [c]), thus becomes the Dual Primality Theorem of Congruence and the Primality Theorem of Compcongruence, and in particular its corollary (1.4.3 [c]), thus becomes the Dual Primality Theorem of Compcongruence.

2 The Dual Primality Theorem of Congruence

Enunciation 2.1 *$\forall N_0, n_0 \in N$ and of equal equality with $N_0 \geq 10$, $0 \leq n_0 < N_0 - p_{max}$, with $\mathbb{P}(\sqrt{(N_0)})$ set of odd prime numbers $\leq \sqrt{(N_0)}$ and with p_{max} prime number higher than $\mathbb{P}(\sqrt{(N_0)})$, a necessary and sufficient condition for $N_0 - n_0$ to be twice a prime number is that $n_0 \not\equiv N_0 \pmod{p_i}$ $\forall p_i \in \mathbb{P}(\sqrt{(N_0)})$ e che $n_0 \neq N_0 - 2^k$.*

Dim. The proof, like the statement, is similar to that of the First Congruence Primality Theorem with the difference that since N_0 and n_0 both even their difference 2*m will always be even and if n_0 is not equal to $N_0 - 2^k$ and is incongruent with N_0 for all odd prime numbers $\leq \sqrt{(N_0)}$ it means that m is a prime number.

3 The Dual Primality Theorem of Compcongruence

Enunciation 3.1 *$\forall N_0, n_0 \in N$ and of equal equality with $N_0 > 2$; $0 \leq n_0 < N_0 - 1$, with $\mathbb{P}(\sqrt{(2N_0)})$ set of odd prime numbers $\leq \sqrt{(2N_0)}$, a necessary and sufficient condition for $N_0 + n_0$ to be twice a prime number is that $n_0 \nparallel N_0 \pmod{p_i}$ $\forall p_i \in \mathbb{P}(\sqrt{(2N_0)})$ e che $n_0 \neq 2^k - N_0$.*

Dim. The proof, like the statement, is quite similar to that of the primality theorem of Compcongruence with the only difference being that since N_0 and n_0 both odd their difference

2*m will always be even and if n_0 is not equal to $2^k - N_0$ and is incompcongruent with N_0 for all odd prime numbers $\leq \sqrt{(2N_0)}$ it means that m is a prime number.

Using the congruence primality theorem and the dual primality theorem of Compcongruence and considering the two intervals [0, (D-1)/2] and [(D+1)/2, D] with D any odd number greater than 5, it is possible to prove Lemoine and Levy's conjecture by referring to the proof of Goldbach's conjecture [d].

Appendix A depicts an example diagram of the two intervals with D=61 in which the numbers n_0 not congruent with (D-1)/2 and the numbers n_0 not congruent with (D+1)/2 and among these the numbers n_0 that are not congruent with (D-1)/2 and together are not compcongruent with (D+1)/2 and which when subtracted and added to (D-1)/2 and (D+1)/2 respectively give rise to the p and 2q terms of Lemoine and Levy's conjecture.

4 Lemoine and Levy's Prime Number Theorem

Enunciation ∀ *D, n_0* ∈ *N and of equal parity with D > 5, 0 ≤ n_0 ≤* $((\mathrm{D}-1)/2)$ *- p ,max with pmax prime number higher than* $\mathbb{P}\left(\sqrt{(\mathrm{D}-1)}\right)$*, where* $\mathbb{P}\left(\sqrt{(\mathrm{D}-1)}\right)$ è *l'set of odd prime numbers* $\leq\sqrt{(\mathrm{D}-1)}$*, a necessary and sufficient condition for* $((\mathrm{D}-1)/2)$ *- n_0 is prime and* $((\mathrm{D}+1)/2)$ *+ n_0 is equal to twice a prime number is that n_0* $\forall p_i \in \mathbb{P}\left(\sqrt{(\mathrm{D}-1)}\right)$ *is incongruous with* $((\mathrm{D}-1)/2)$ *ed incongruous with* $((\mathrm{D}+1)/2)$.

Dim. From the First Congruence Theorem (Corollary 1,2,3 [c]) and the Dual Primality Theorem of Compcongruence (3), placing the most restrictive conditions between the two, it follows that there is certainly at least one n_{01} incongruity with $((\mathrm{D}-1)/2)$ and at least one n_{02} incompcongruent with $((\mathrm{D}+1)/2)$ but we cannot derive from them that esiste anche un $n_0 = \mathrm{n}_{01} = \mathrm{n}_{02}$.

Lemma 4.1 *There exists an infinite set of numbers D such that (D-1)/2 and (D+1)/2 already satisfy the conjecture, one of them being equal to p and the other to 2*q.*

Dim. For D ≥ 15 the (D-1)/2 and (D+1)/2 equal to 2q can never be multiples of 3 and therefore in the triads (D-1)/2, (D+1)/2 and (D+1)/2 + 1 (e.g.: 7,8,9 - 8,9,10 - 9,10,11 - etc.) in which one of the two (D ± 1)/2 is equal to 2q and (D+1)/2 + 1 is a multiple of 3 the other (D ± 1)/2 will be irregularly equal to p. We denote these numbers Dpq.

To prove Lemoine and Levy's conjecture, on the other hand, it must be established that for every D > 5 there is at least un numero $\mathrm{n}_0 = \mathrm{n}_{01} = \mathrm{n}_{02}$ such that $((\mathrm{D}-1)/2)$ *- n_0 is prime,* $((\mathrm{D}+1)/2)$ *+ n_0 is the double of a prime and that their sum is equal to D.* We denote this n_0 as the number LL (Lemoine Levy) *di D*.

To this end, we resort to Lemoine and Levy's study of the density of LL numbers.

5 The density of prime numbers LL of Lemoine and Levy

Let us say at the outset that each LL number must have the following characteristics:

- its modulus class 2 must be equal to 1;
- its modulus classes of successive primes (3, 5, 7, etc.) less than or equal to the $\left(\sqrt{((\mathrm{D}-1)/2)}\right)$ must not be equal to the classes corresponding to the remainder of $((\mathrm{D}-1)/2)$ (for non-congruence) and to the complement of the remainder of $((\mathrm{D}+1)/2)$ (for non-congruence) for the same modules.

Having said this, let us see how to calculate the number of LL less than a $((D-1)/2) \geq$ 121, a condition arising as we know (study of Goldbach's conjecture, 1.7.1) from the necessity that D-1 belongs to the interval [0, $\sqrt{(\mathbf{D-1})}$ #].

Having then selected any D ≥ 243, we call pmax the highest prime number less than or equal to the $\sqrt{(D-1)}$. Let us then consider the table-interval of the natural numbers [0, pmax#] where pmax# is the prime of pmax and corresponds to the product 2*3*5*.......* pmax, a product that corresponds to the last number of the relevant number-class table pmax (study of Goldbach's conjecture , 1.5.1) of biunivocal correspondence between the numbers of the interval and their respective combinations of congruence classes.

Let us now eliminate from this table [0, pmax#] each of the rows that has a congruence class mod 2 equal to 0 and/or congruence classes of the following modules (3, 5, ... , pmax) equal to the classes corresponding to the remainder of $((D-1)/2)$ and the complement of the remainder of $((D+1)/2)$ for the same modules.

The M-numbers in the table, which were not eliminated through the previous sieve, can then only be:

a) those which in the Number-Class Table pmax have in their corresponding combination of congruence classes the congruence class 1 modulo 2
b) those which in the number-class table pmax for each odd p_i belonging to the set $\mathbb{P}\left(\sqrt{(\mathbf{D-1})}\right)$ have in their corresponding combination of congruence classes one of the p_i - 2 possible congruence classes of modules 3, 5, ... , p_{max} i.e. excluding the classes corresponding to the remainder of $((D-1)/2)$ (for non-congruence) and the complement of the remainder of $((D+1)/2)$ (for non-congruence) for the same modules.

For each number D then the rows (combinations of classes) of the table [0, pmax#] not deleted and thus the M incongruous with (D-1)/2 and incompcongruous with (D+1)/2, according to the combinatorial calculation and thus corresponding, will be:

(5.1) $\prod_{p=3}^{p_{max}}(p-2)$

and their density in the interval $[0, \sqrt{(\mathbf{D-1})}$ #] will be:

(5.2) $Dncncomp_{[0,\sqrt{(\mathbf{D-1})}\,\#]} = \frac{\prod_{p=3}^{pmax}(p-2)}{\prod_{p=2}^{pmax} p} = \frac{1}{2} * \prod_{p=3}^{pmax} \frac{(p-2)}{p}$

The value of $p_{\max}$ of (5.2), and thus also the density $Dncncomp_{[0,\sqrt{(\mathbf{D-1})}\,\#]}$, remains the same for each interval $[0, \sqrt{(\mathbf{D-1})}$ #] with D such that it results $p_{\max} < \sqrt{D-1} < p_{\max succ}$ where $p_{\max}$ is the highest prime less than $\sqrt{D-1}$ e $p_{\max succ}$ the prime number immediately following $p_{\max}$, Then for all Ds belonging to the interval $[p_{max}{}^2 + 1, p_{\max succ}{}^2]$ the density $Dncncomp_{[0,\sqrt{(\mathbf{D-1})}\,\#]}$ remains the same and equal to that of the numbers Dpq contained in it.

To the density $Dncncomp_{[0,\sqrt{(\mathbf{D-1})}\,\#]}$ which represents the density of the numbers M of the interval $[0, \sqrt{(\mathbf{D-1})}$ #] incongruous with (D-1)/2 and incompcongruous with (D+1)/2 for only p-modules$_i$ belonging to the set of odd prime numbers $\mathbb{P}\left(\sqrt{(\mathbf{D-1})}\right)$ corresponds to a density $Dncncomp_{[0,\ (D-1)/2]}$ of the numbers incongruous with (D-1)/2 and incompcongruous with (D+1)/2 in the interval [0, (D-1)/2] always only for the p modules$_i$ belonging to the set of odd prime numbers $\mathbb{P}\left(\sqrt{(\mathbf{D-1})}\right)$ (and therefore not for the modulus 2), namely the numbers LL of D. But for the latter we already know (Lemma 4.1) that there exists at least one number LL = 0 that satisfies the conjecture and that allows us to write

(5.3) $Dncncomp_{[0,\ (D-1)/2]} \geq \frac{2}{D-1}$

Now since the density $Dncncomp_{[0,\sqrt{(\mathbf{D-1})}\ \#]}$ equal for all Ds belonging to the interval $[p_{max}{}^{2} + 1, p_{\max succ}{}^{2}]$ will also be equal the corresponding density $Dncncomp_{[0,\ (D-1)/2]}$ of the same D and therefore (5.3) is valid for every D of the said interval.

And again, multiplying both members of (5.3) by (D-1)/2 the number M_{LL} of the LL numbers of D will be :

(5.4) $M_{LL} = Dncncomp_{[0,\ (D-1)/2]} * \frac{D-1}{2} \geq 1$

Consequently M $_{LL}$ will always be greater than or equal to 1 and thus there will always be at least one pair of primes p and q such that p+2q = D as predicted by Lemoine and Levy's conjecture.

For Ds less than 243, Lemoine and Levy's conjecture is easily verifiable.

APPENDIX A

To give a numerical example, let us assume D = 61 and for ease of exposition
We denote the interval [0, D] with the two intervals [0, (D-1)/2] = [0, 30] and [(D+1)/2, D] = [31, 61] in the manner indicated below.
Lateral to the two intervals are the numbers n_0 of the interval
[0, 30] not congruent (nc) with 30 and the numbers n_0 of the interval [31, 61] not compcongruent (ncc) with 31

n_0		[0, (D-1)/2]	[(D+1)/2,0]	
0		30	31	ncc
1	nc	29	32	
2		28	33	
3		27	34	ncc
4		26	35	
5		25	36	
6		24	37	
7	nc	23	38	ncc
8		22	39	
9		21	40	
10		20	41	
11	nc	19	42	
12		18	43	
13	nc	17	44	
14		16	45	
15		15	46	ncc
16		14	47	
17	nc	13	48	
18		12	49	
19	nc	11	50	
20		10	51	
21		9	52	
22		8	53	
23	min. of rad(61)	7	54	
24		6	55	
25	min. of rad(61)	5	56	
26		4	57	
27	min. of rad(61)	3	58	ncc
28		2	59	
29		1	60	
30		0	61	

In the table there are 6 nc numbers in the interval [0, 30] and 5 ncc numbers in the interval [31, 61] and only one n_0 equal to 7 nc and ncc. (p=23 and q=19)

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