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Investigation of Collatz's conjecture

Abstract

In the article, a study is developed on the Collatz conjecture that is able to determine for each initial odd natural number the relationship existing between the even and odd steps of the relative orbit, to demonstrate the narrowness of the succession and the absence in it of cycles preceding that 4, 2, 1.

1 Introduction

The Collatz conjecture (also known as the 3n + 1 conjecture) is a mathematical conjecture that is still unsolved. It was first stated in 1937 by Lothar Collatz, after whom it is named.

The conjecture can be expressed with the following algorithm:

1. Take a positive integer n.
2. If n = 1, the algorithm terminates.
3. If n is even, divide by two; otherwise multiply by 3 and add 1.

i.e. algebraically:

$$f(n) = \begin{cases} \frac{n}{2} \; se\ n\ \text{è}\ pari \\ 3n+1 \; se\ n\ \text{è}\ dispari \end{cases}$$

&{{\even}{end{cases}}this function repeatedly, it is possible to form a succession $\{a_n\}_{n\,\mathcal{E}\,N}$, also called orbit or trajectory, which has as its first element any positive integer n and as its subsequent elements those obtained by applying the function to the previous element, i.e:

$$a_i = \begin{cases} n \;\; per\ i = 0 \\ f(a_{i-1}) \;\; per\ i > 0 \end{cases}$$

For example, starting with n = 18, we obtain the sequence 18, 9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20. 10, 5, 16, 8, 4, 2, 1

The Collatz conjecture asserts that this succession always arrives at an element equal to 1 regardless of the starting value. More formally:

$\forall\, n\,\mathcal{E}\,N > 0 \;\; \exists\, i\,\mathcal{E}\,N: (a_0 = n \;\; => \;\; a_i = 1)$

2 The state of research in a nutshell

The conjecture has been verified by computer for all values up to about 10^{20} , which of course does not mean that it has been proven.

On the basis of probabilistic considerations, if one considers only the odd numbers in the sequence $\{a_n\}_{n\,\mathcal{E}\,N}$one can state that on average the next odd number should be about 3/4 of the previous one, which suggests that they gradually decrease until they reach 1.

In 1976 the mathematician Riho Terras demonstrated that, *for almost all numbers*, after the application of an appropriate number of steps the sequence reaches a value smaller than the starting value.

Finally, mathematician Terence Tao has recently demonstrated that for *almost all numbers*, the sequence sooner or later reaches a value much lower than the initial N, such as N/2 or the logarithm of N or any function *f(N)* tending to infinity with N. Obviously, this demonstration increases the belief that the conjecture is true.

3 The set of terms of the succession

Let us now analyse the various terms of the sequence $\{a_n\}_{n \varepsilon N}$ and let us assume that the succession arises from any odd number d_0 (in fact, if the initial number is even and different from a power of 2, applying the function f(n) for even n, one always arrives at an odd number d_0 different from 1, while if the initial number is equal to a power of 2, applying f(n) one arrives directly at 1). Assuming then that $a_0 = d_0$ and knowing that the terms of the succession are of the type 3d+1 (with d being an odd number) or of the type $(3d+1)/2^h$ or finally of the type d (odd integer), we can state that all the terms of the succession belong to the following grid:

3*1+1	3*3+1	3*5+1	3*7+1	3*9+1	3*11+1	3*(..)+1
(**3*1+1**)/2	(**3*3+1**)/2	(**3*5+1**)/2	(**3*7+1**)/2	(**3*9+1**)/2	(**3*11+1**)/2	(3*(**..**)+1)/2
(**3*1+1**)/2^h =p	(**3*3+1**)/2^h =p	(**3*5+1**)/2^h =p	(**3*7+1**)/2^h =p	(**3*9+1**)/2^h =p	(**3*11+1**)/2^k =p	(3*(**..**)+1)/2^h =p
....	....	....	....	....	....	....
(**3*1+1**)/2 =d^k	(**3*3+1**)/2 =d^k	(**3*5+1**)/2 =d^k	(**3*7+1**)/2 =d^k	(**3*9+1**)/2 =d^k	(**3*11+1**)/2 =d^k	(3*(**..**)+1)/2 =d^k

in which the first row contains the infinite **even terms** 3d+1, the second row the infinite **even terms** (3d+1)/2, the intermediate rows ***the*** infinite **even** terms $(3d+1)/2^h$ and the final row the infinite **odd** terms $(3d+1)/2^k$ with 2^k the highest power of two that divides (3d+1). Logically, where 4 does not divide (3d+1) there will be no intermediate terms in the corresponding column and the final one, equal to the second term, will be precisely (3d+1)/2.

Remark 3.1 We immediately observe that for any d_i if $3d_i$ +1 is divisible only by two, resulting in an odd number d_j the next term of the sequence $3d_j$ +1 moves to a column of the grid following that of $3d_i$ +1 (e.g. **3*7+1**); on the other hand, if $3d_i$ +1 is divisible by 2^k with k > 1 then $3d_j$ +1 moves to a column of the grid preceding that of $3d_i$ +1 (e.g. **3*9+1**). Obviously if the first case (divisibility of $3d_i$ +1 by 2^k only for k=1) is repeated with successive $3d_j$ +1 for m times there will be m forward shifts of the term $3d_i$ +1 and vice versa in the sense that if $3d_i$ +1 is divisible by 2^k with k > 1 the resulting term $3d_i$ +1 will move in the grid the further back the greater k is.

This observation allows us to emphasise the non-monotonic character of our succession and the centrality that, for our study of the conjecture, the term $(3d+1)/2^k$ with 2^k the highest power of two that divides (3d+1), has.

4 An almost equivalent succession

For the purposes of our demonstration, we can replace our succession $\{a_n\}_{n \varepsilon N}$ with the sequence $\{d\}_{n \varepsilon N}$ which has as its first element any positive odd integer d_0 and as its subsequent elements the odd numbers obtained by applying to the previous element d the function g(d) in which 2^k is the greatest power of two that divides 3d+1:

(4.1) $g(d) = \frac{3d+1}{2^k}$

and therefore each element of the succession $\{d_n\}_{n \,\varepsilon\, N}$ sarà:

$$(4.2)\ \ d_i = \begin{cases} d_0 \; per \; i = 0 \\ g(d_{i-1}) \;\; per \; i > 0 \end{cases}$$

The orbit of the succession $\{d\}_{n \,\varepsilon\, N}$, with equal input d_0 , traverses all the odd-numbered nodes of the orbit of the succession $\{a_n\}_{n \,\varepsilon\, N}$.

As it exists for the succession $\{a_n\}_{n \,\varepsilon\, N}$ also for the quasi-equivalent $\{d\}_{n \,\varepsilon\, N}$ la congecture would be false if there existed a number d_0 Ɛ N for which the two successions did not contain the number 1, i.e. if the successions were unlimited above or if there existed in them a cycle that repeats itself without ever giving 1. The truth or otherwise of the conjecture for the $\{d\}_{n \,\varepsilon\, N}$ also involves that of the $\{a\}_{n \,\varepsilon\, N}$ in that if the first is superiorly limited, the second will also be limited (and vice versa) and if there exists an even number [(**3*d+1**) or (**3*d+1**)/2^h] that repeats itself in the succession $\{a\}_{n \,\varepsilon\, N}$ giving rise to a cycle that repeats itself without ever giving 1 there will surely also be an odd number [**d** or (**3*d+1**)/2^k] that will repeat itself in the sequence. $\{d\}_{n \,\varepsilon\, N}$.

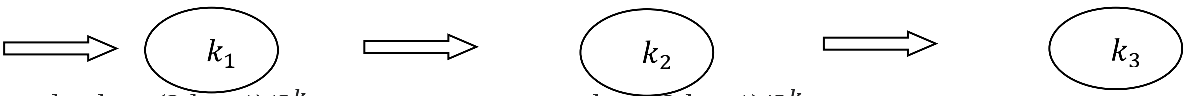
Then to prove that the succession $\{d\}_{n \,\varepsilon\, N}$ always arrives at 1, we must prove that:

a) the succession is superiorly limited and therefore there is no odd number d for which the succession tends to infinity

b) up to the node of the orbit equal to 1 there are no numbers in it that repeat without ever giving 1 and therefore no term in the succession $\{d_n\}_{n \,\varepsilon\, N}$ is equal to one of its preceding terms.

5 Increments and Decrements in Orbit $\{d_n\}_{n \,\varepsilon\, N}$

In the equivalent (orbit) succession $\{d_n\}_{n \,\varepsilon\, N}$ each node k_i has as input the output d_{i-1} of the previous node k_{i-1} and as output the number $d_i = (3^* d_{i-1} + 1)/2^{k_i}$ with 2^{k_i} the highest power of 2 dividing $(3^* d_{i-1} + 1)$.



d_0 $d_1 = (3d_0 + 1)/2^{k_1}$ $d_2 = (3d_1 + 1)/2^{k_2}$

Analysing the behaviour of the even numbers (3d+1) ∀ d Ɛ D, with D being the set of odd positive integers, it can be observed that the d numbers generating (3d+1) divisible by 2^k only with k=1 are of the type:

(5.1) d = 3 + 4m with m = 0, 1, 2, 3, 4, ...

and thus belong to the set D_1 = {3, 7, 11, 15, 19, ...}

and those generating (3d+1) divisible by 2^k with k>1 are of the type:

(5.2) d = 1 + 4m with m = 0, 1, 2, 3, 4, ...

and thus belong to the set D_2 = {1, 5, 9, 13, 17, 21, ...}

Remark 5.3 The density in D of the numbers belonging to the set D_1 is equal to 1/2 as is the density of the numbers belonging to D_2 . In particular, we have that the densities in D_2 (with k≥2), of the numbers d generating (3d+1) divisible by 2^k are equal to $1/2^{k-1}$.

Now for each node K_i of the orbit between the output d_i and the input d_{i-1} the following approximate relationship exists:

(5.4) $d_i = (3d_{i-1} +1)/2^{k_i} \approx 3d /_{i-1}2^{k_i}$

from which it follows that if $K_i = 1$ (see also Remark 3.1) and therefore d_{i-1} belongs to D_1 , $d_i > d_{i-1}$ and holds:

(5.5) $d_i \approx (3/2) * d_{i-1}$

with a ratio between d and $d_{i\ i-1}$ greater than 1, which we will call the **increment ratio**.

On the other hand, if $K_i > 1$ and therefore d_{i-1} belongs to D_2 , then $d_i < d_{i-1}$ applies:

(5.6) $d_i \approx (3/2^{k_i}) * d_{i-1}$

with a ratio between d and $d_{i\ i-1}$ less than 1, which we will call the **decrement ratio.**

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Observation 5.7 It should be noted that if d_{i-1} passes through a node $K_i = 1$, its value increases by ½, if it passes through a node $K_i = 2$, its value decreases by ¼ (less than the increase with $K_i = 1$), and if it passes through a node $K_i > 2$, the value of d_{i-1} decreases by $(2^{k_i}\text{-}3)/2^{k_i}$ (greater therefore than the increase with $K_i = 1$*)*

So if, for example, the orbit relative to any input d_{in} consisted of a periodic sequence of nodes consisting of: $K_1 =1$, $K_2 =2$, $K_3 =1$ and $K_4 =3$ the input d_{in} at the node K_1 would first increase to $(3/2)*d_{in}$, then decrease to $(9/8)*d_{in}$, then increase again to $(27/16)*d_{in}$ and finally to decrease to $(81/129)*d_{in}$; at the end of the sequence the output d_{out} from node K_4 would be smaller than the input d_{in} at node K_1 and would be worth approximately $0.63* d_{in}$.

If instead the orbit consisted of a periodic sequence of nodes consisting of: $K_1 =1$, $K_2 =1$, $K_3 =2$ and $K_4 =2$ the input d_{in} to the node K_1 would first increase to $(3/2)*d_{in}$, then to $(9/4)*d_{in}$, then decrease to $(27/16)*d_{in}$ and then to $(81/64)*d_{in}$; at the end of the sequence the output d_{out} from node K_4 would be greater than the input d_{in} at node K_1 and would be worth approximately $1.26* d_{in}$.

Obviously these examples are only illustrative and tell us nothing about the actual course of an orbit generated by d_0 , the sequence of nodes K_i as well as their value (1,2,3,4,etc.); the course of the orbit in fact derives only from d_0 and the succession $\{d_n\}_{n\,\varepsilon\,N}$ that it determines by iteratively applying the function g(d).
Now for the purposes of our study it is of interest to establish whether the succession $\{d_n\}_{n\,\varepsilon\,N}$ is superiorly bounded or not, and thus whether the increasing or decreasing ratios prevail in the orbit or whether they are equivalent.

Observation 5.8 From what has been written above (Observation 5.7), nodes with k=1 (i.e. $d_i \,\varepsilon\, D_1$) result in an increasing ratio while nodes with k>1 (i.e. $d_i \,\varepsilon\, D_2$) result in a decreasing ratio as k increases. Furthermore, in order to evaluate the relative variation at d_0 of any d_i of the orbit of d_0 one must also consider the possible sequences of consecutive nodes with k=1, which, if they were uninterrupted, would determine the divergence of the succession and thus its upper boundlessness.

6 Average increase and decrease ratios in the orbit $\{d_n\}_{n\,\varepsilon\,N}$

In order to determine the average variation relative to d_0 of any d_i of the orbit of d_0 , one must therefore refer to the average value of the increment ratios determined by the sequences of nodes with k=1 present in the current orbit and the average value of the decrement ratios determined by the nodes with k>1 present in it.

To this end, let us consider an A.I. computer algorithm that simulates the orbit of d_0 described by the $\{d_n\}_{n \varepsilon N}$ with the exclusion of the intermediate odd nodes (numbers) of the sequences of consecutive nodes with k=1. This is its flowchart:

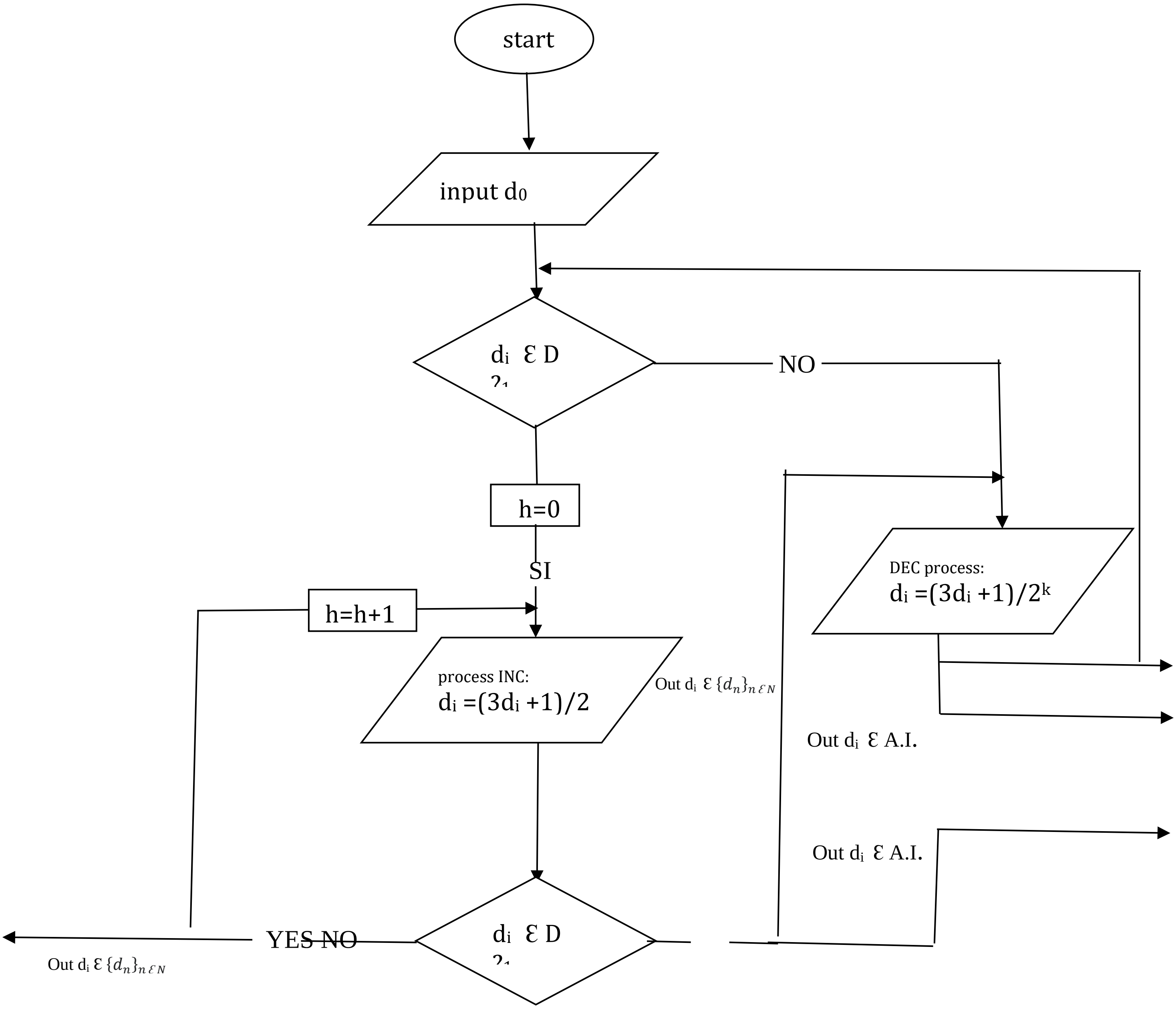

Figure 1

Four outputs are shown in the flowchart of the algorithm, two give rise to all nodes of the succession $\{d_n\}_{n \varepsilon N}$ while the other two relate to the outcomes of the complete DEC (decrement) and INC (increment) processes (i.e. up to d_i Ɛ D_2), which determine the decrements and increments of d_i respectively. Obviously, for the purpose of proving the conjecture, we know that d_i (starting from d_0) will go through either the INC process or the DEC process depending on whether the same d_i belongs to D_1 or D_2 while we know nothing about the value of the initial d_0 and the d_i of the orbit of d_0 .

6.1 Mean decrement ratio theorem of an orbit

Enunciation *The average decrement ratio determined by the m_2 odd natural numbers d_i of the orbit, produced by any odd natural number d_0 , and generating $(3d_i +1)$ divisible by 2^k with $k >1$ tends to 0.375 as m_2 increases.*

Demonstration If we assume, just for the sake of convenience of reasoning, that the d_i of the orbit of d_0 belonging to D_2 are all included in the interval [1, 2^n] so that all the d_i of the orbit belong to the subset of D_{2n} :

(6.2) $D_{2n} = \{1, 5, 9, 13, 17, 21, ... 2^n +1\}$

we can calculate the average decrement ratio $v_{md(n)}$ that a d_i of the set D_{2n} undergoes in the DEC process by making a weighted geometric average (assuming the respective densities as weights) of the decrement ratios that all the d_i numbers belonging to the set D_{2n} undergo along the orbit:

$$(6.3)\quad v_{md}(n) = \sqrt[M]{\prod_{k=2}^{n} x_k{}^{y_k}}$$

where x_k is the decrement ratio equal to $3/2^k$; y_k is the density (Remark 5.3) of the numbers d_i Ɛ D_{2n} generating ($3d_i$ +1) divisible by 2^k which is equal to $1/2^{k-1}$ and M is the sum of the densities equal to $\sum_{k=2}^{n} \frac{1}{2^{k-1}}$.

Therefore, (6.3) becomes:

$$(6.4)\quad v_{md}(n) = \sqrt[\sum_{k=2}^{n}\frac{1}{2^{k-1}}]{\prod_{k=2}^{n} \left(\frac{3}{2^k}\right)^{\frac{1}{2^{k-1}}}}$$

Obviously, however, the d_i of the orbit of d_0 belonging to D_2 can span, from d_0 , the entire infinite set D_2 and thus the average decrement ratio v_{md} (D_2) will be equal to:

$$(6.5)\quad v_{md}(\mathrm{D2}) = \lim_{n\to\infty}\left(v_{md}(n)\right) = \lim_{n\to\infty} \sqrt[\sum_{k=2}^{n}\frac{1}{2^{k-1}}]{\prod_{k=2}^{n} \left(\frac{3}{2^k}\right)^{\frac{1}{2^{k-1}}}} = 0{,}375$$

Obviously, the longer the orbit of d_0 , and with it the number m_2 of the d_i Ɛ D_2 (d_i all distinct from each other as we will demonstrate later) that pass through the DEC process, the closer the average of the decrement ratios that have occurred approaches the calculated mean value of (6.5) as we will demonstrate later.

6.6 Theorem of the average increment ratio of an orbit

Enunciation *The average increment ratio determined by the m_1 odd natural numbers d_i of the orbit, produced by any odd natural number d_0 , and generating sequences of h ($3d_i$ +1) divisible consecutively by 2^k with k =1 tends to 2.250 as m_1 increases.*

Demonstration If we assume, just for the sake of convenience of reasoning, that the d_i of the orbit of d_0 belonging to D_1 are all included in the interval [1, 2^n] so that all the d_i of the orbit belong to the subset of D_{1n} :

(6.7) $D_{1n} = \{3, 7, 11, 15, 19, ... 2^n -1\}$

we can calculate the average increment ratio $v_{mi(n)}$ that a d_i of the set D_{1n} undergoes in the INC process by making a weighted geometric average (assuming the respective densities as weights) of the increment ratios that all the d_i numbers belonging to the set D_{1n} undergo along the orbit.

Remark 6-8 In order to determine the densities of the different increment ratios, dependent on the length h of the sequences of the INC process traversed by the generic d_i of the set D_{1n} , we analyse the Figure in Annex A. In it we can see that the densities in D_{1n} of the sequences of the INC process and thus of the increment ratios relative to a d_i Ɛ D_{1n} are equal for sequences of length h (with h=1, 2, 3, ...n-2) to $1/2^h$ and for h=n-1 to $1/2^{(n-2)}$.

Lemma 6.9 *From the study of figure 2 and what we have already deduced from it, we can state that relative to an interval [1, 2^n], the maximum length of an INC sequence determined by a d_i of the set D_{1n} belonging to the aforementioned interval is equal to n-1, after which there will certainly be a decrease in the orbit.*

Based on the densities in D_{1n} of the INC process sequences determined above and the increment ratio of d_i of equal to 3/2 [see (5.5)] for each individual step in the INC process, we can now calculate the average relative increment ratio $v_{mi(n)}$ that a d_i of the set D_{1n} undergoes in a sequence of INC processes by making a weighted geometric average (assuming the respective densities of the different sequences as weights) of the increment ratios that all the d_i numbers belonging to the set D_{1n} undergo along the orbit:

$$(6.10) \quad v_{mi}(n) = \sqrt[M]{\prod_{h=2}^{n} {x_h}^{y_h}}$$

where x_h is the increment ratio equal to $(3/2)^h$; y_k is the density (Remark 6.8) of the numbers d_i Ɛ D_{2n} generating ($3d_i$ +1) divisible by 2 which is equal to $1/2^h$ and M is the sum of the densities equal to $\sum_{h=1}^{n} \frac{1}{2^h}$.

Therefore (6.10) becomes:

$$(6.11) \quad v_{mi}(n) = \sqrt[\sum_{h=1}^{n-2}\frac{1}{2^h}]{\prod_{h=1}^{n-2}\left[\left(\frac{3}{2}\right)^h\right]^{\frac{1}{2^h}} * \left[\left(\frac{3}{2}\right)^{n-1}\right]^{\frac{1}{2^{n-2}}}}$$

Obviously, however, the d_i of the orbit of d_0 belonging to D_1 can span, from d_0 , the entire infinite set D_2 and thus the average decrement ratio v_{mi} (D_2) will be equal to:

$$(6.12) \quad v_{mi}(\mathrm{D1}) = \lim_{n\to\infty}\left(v_{mi}(n)\right) = \sqrt[\sum_{h=1}^{n-2}\frac{1}{2^h}]{\prod_{h=1}^{n-2}\left[\left(\frac{3}{2}\right)^h\right]^{\frac{1}{2^h}} * \left[\left(\frac{3}{2}\right)^{n-1}\right]^{\frac{1}{2^{n-2}}}} = 2.250$$

Obviously, the longer the orbit of d_0 , and with it the number m_1 of the d_i Ɛ D_1 (d_i all distinct from each other as we will demonstrate later) passing through the sequences of the INC process, the closer the average of the increment ratios occurring approaches the calculated mean value of (6.12) as we will demonstrate later.

7 The upper boundary of the orbit $\{d_n\}_{n\,\varepsilon\,N}$

The Out d_i of the Computer Algorithm in figure 1 generates a succession (orbit) $\{AI\}_{n\,\varepsilon N}$ of d_0 that equals the orbit of d_0 described by the $\{d_n\}_{n\,\varepsilon N}$ with the exclusion of the intermediate odd nodes (numbers) of the sequences of nodes with k=1 of the INC process. This implies that the upper or lower bound of the $\{AI_n\}_{n\,\varepsilon N}$ also implies that of the $\{d\}_{n\,\varepsilon N}$ and of the $\{a_n\}_{n\,\varepsilon N}$.

7.1 Upper limit theorem of the Collatz orbit

Enunciated *The Collatz orbit $\{d_n\}_{n\,\varepsilon N}$ is limited superiorly*

Demonstration Given that in the computer algorithm in Figure 1:

- the outputs d_{i2} Ɛ D_2 are greater than the outputs d_{i1} Ɛ D_1 because, with the same density (1/2) in D, the INC process crossed by a d_{i1} always generates a d_{i2} while the DEC process crossed by a d_{i2} generates both d_{i1} and d_{i2}
- While there are d_{i2} such that $3d_{i2}+1$ is equal to 2^n (with n even) and thus generates downstream of the DEC process the number 1 (term of the Collatz orbit), there are no d_{i1} such that the INC process activates a sequence of infinite increments since according to Lemma 6.9 the maximum length of an INC sequence determined by a d_{i1} is equal to n-1 odd steps with n equal to the exponent of the first power of 2 greater than d_{i1}
- For any d_i of a succession (orbit) $\{AI\}_{n\,\varepsilon N}$ generated by an odd number d_0 the following relationship applies:

  (7.2) $\frac{d_i}{d_0} = r_{1*}\, r_{2*}\, r_{3*}\, r\, r_{4*5*}\, r_{6*}$.. r_j

  in which the second term constitutes a distribution generated by the function g(d) in which the r_i (with i<j) indicate the ratios d /d_{jj-1} of increase and decrease that the various d_i of the orbit undergo in the respective INC and DEC processes (each INC process is always followed by at least one DEC process, but there can also be more since in each orbit d_{i2} Ɛ D_2 is always greater than or equal to d_{i1} Ɛ D)$_1$

Let us assume that there exist one or more numbers d_0 , generators of the succession $\{AI_n\}_{n\,\varepsilon\,N}$and thus also of the successions $\{d\}_{n\,\varepsilon\,N}$ and $\{a_n\}_{n\,\varepsilon\,N}$, which give rise in the INC processes to ratios of increase of d_i greater than V_{mi} (D_1) and in the DEC processes to ratios of decrease of d_i less than V_{md} (D_2) with the consequence of having a succession of averagely increasing values of d_i even if not infinite thanks to Lemma 6.9.

But this increasing trend in the succession cannot continue indefinitely. As the number of d_i (i.e. the number of nodes in the orbit of d_0) increases, the mean values of the increment and decrement ratios occurring approach the two mean values calculated with (6.5) and (6.12) so that each new d_{i1} gives rise to an average overall increase ratio with respect to d_0 of the INC processes closer and closer to 2.25 and each new d_{i2} gives rise to an average overall decrease ratio with respect to d_0 of the DEC processes closer and closer to 0.375 with the result that the more the orbit lengthens without meeting 1 (as the last node of the conjecture) the more the overall increases and decreases cancel each other out.

In fact, (7.2) can also be written as:

(7.3) $\frac{d_j}{d_0} = \prod_{i=1}^{Rinc} r_{in} * \prod_{d=1}^{Rdec} r_{de}$

where r_{in} and r_{de} are the individual ratios of increment and decrement and Rinc and Rdec (with Rdec≥Rinc) are the number of increment ratios and the number of decrement ratios present in the orbit from d_0 and d_j .
The main property of the mean value of a geometric distribution is:

$$\prod_{i=1}^{n} x_i = \prod_{i=1}^{n} x_{me}$$

where x_i is the generic term of a distribution of n terms and x_{me} its mean value.

If, on the other hand, we limit ourselves to the product of only m (with m << n) terms of the distribution, it will always be very likely:

$$\prod_{i=1}^{m} x_i \neq \prod_{i=1}^{m} x_{me}$$

with the difference between the two products decreasing as m increases until it reaches zero for m=n or, in the case of infinite distribution, until equality:

$\lim_{m\to\infty} \prod_{i=1}^{m} x = \lim_{m\to\infty} \prod_{i=1}^{m} x_{me}$

So if we refer to our distribution (7.2) { r_{1*} r r_{2*3*} r_{4*} r_{5*} r_{6*}......... } of infinite terms belonging to the interval]0, ∞] of R we can write by analogy:

(7.4) $\lim_{n\to\infty} \prod_{i=1}^{n} r_i = \lim_{n\to\infty} \prod_{i=1}^{n} r_m$ with r_m mean value of the distribution (7.2)

and according to (7.3):

(7.5) $\lim_{n\to\infty} \prod_{i=1}^{n} r_i = \lim_{n\to\infty} \prod_{i=1}^{Rinc} r_{in} * \prod_{d=1}^{Rdec} r_{de} = \lim_{rinc\to\infty} \prod_{i=1}^{Rinc} r_{in} * \lim_{rdec\to\infty} \prod_{d=1}^{Rdec} r_{de} =$

$= \lim_{rinc\to\infty} \prod_{i=1}^{Rinc} v_{mi}(D_1) * \lim_{rdec\to\infty} \prod_{d=1}^{Rdec} v_{md}(D_2) = \lim_{rinc\to\infty} v_{mi}(D_1)^{Rinc} * \lim_{rdec\to\infty} v_{md}(D_2)^{Rdec}$

and being $v_{mi}(D_1) = 2{,}250$, $v_{md}(D_2) = 0.375$ so $v_{mi}(D_1) * v_{md}(D_2) = 0.84375$ and being among other things along the orbit Rdec ≥ Rinc, it will always result that the $\lim_{n\to\infty} \prod_{i=1}^{n} r_i < 1$.

The result is that, as long as the number 1 has not already been generated in the orbit, the output d_i gets closer and closer to the input d_0 and then falls below it. Furthermore, the disparity in the orbit between the d_{i2} and the d_{i1} with the former prevailing (see premise) pushes the orbit downwards and the d_i to take on average values below d_0 .

Observation 7.6 Ultimately, the longer the orbit of any d_0 is prolonged, the more the d_i of its advancement is kept smaller than d_0 and in any case the succession $\{AI_n\}_{n\,\varepsilon\,N}$ is superiorly limited, just as the successions will also be superiorly limited $\{d\}_{n\,\varepsilon\,N}$ and $\{a_n\}_{n\,\varepsilon\,N}$.

8 Orbit Steps $\{a_n\}_{n\,\varepsilon\,N}$

Using a more arithmetical approach to the succession $\{d\}_{n\,\varepsilon\,N}$ we can derive the expression of the generic id-th odd term d_{id} of the succession.

Lemma 8.1 *Given a sequence $\{d\}_{n\,\varepsilon\,N}$ generated by any positive odd integer d_0 and having as successive terms the odd numbers obtained by applying to the previous term d the function g(d) = $\frac{3d+1}{2^k}$, where 2^k is the greatest power of two that divides 3d+1, the expression of the generic id-th odd term d_{id} of the sequence is:*

(8.2) $d_{id} = \frac{3^{id}*d_0+3^{id-1}+3^{id-2}*2^{s_1}+3^{id-3}*2^{s_1+s_2}+\cdots.........+3*2^{s_1+\cdots+s_{id-2}}+2^{s_1+\cdots+s_{id-1}}}{2^{s_1+\cdots+s_{id}}}$

where:

- *d_{id} is the generic odd term d_i of the succession resulting from the division of the term $3d_{id-1}$+1 by the greatest power of 2 that can divide it*
- *s_i is the exponent of the highest power of 2 by which the term is divisible $3d_{i-1}$+1*
- *id represents the number of odd-numbered steps in the sequence $\{d\}_{n\,\varepsilon\,N}$ (and of the $\{a\}_{n\,\varepsilon\,N}$) up to the node d_{id}*
- *$s_1 + s_2 + \cdots ......+s_{id}$ = ip represents the sum of the exponents of the greatest powers of two by which the terms are divisible $3d_{i-1}$+1 for all d_i of the sequence $\{d\}_{n\,\varepsilon\,N}$ up to d_{id}*

Demonstration According to the definition of $\{d\}_{n\,\varepsilon\,N}$ the first terms d_{id} (with id index of the terms of the sequence) following d_0 are:

$d_1 = \frac{(3d_0+1)}{2^{s_1}}$; $d_2 = \frac{(3d_1+1)}{2^{s_2}} = \frac{3*\frac{(3d_0+1)}{2^{s_1}}+1}{2^{s_2}} = \frac{3^2*d_0+3+2^{1_1}}{2^{s_1}*2^{s_2}}$; $d_3 = \frac{(3d_2+1)}{2^{s_3}} = \frac{3*\frac{3^2*d_0+3+2^{1_1}}{2^{s_1}*2^{s_2}}+1}{2^{s_3}} =$
$\frac{3^3*d_0+3^2+3*2^{s_1}+2^{s_1}*2^{s_2}}{2^{s_1}*2^{s_2}*2^{s_3}} = \frac{3^3*d_0+3^2+3*2^{s_1}+2^{s_1+s_2}}{2^{s_1+s_2+s_3}} = \frac{3^{id}*d_0+3^{id-1}+3*2^{s_1}+2^{s_1+s_{id-1}}}{2^{s_1+s_2+s_{id}}}$ where id=3

Proceeding in the same way for the terms following the third one, we arrive at (8.2) for each d_{id} .

Remark 8.3 Note that the indices id and ip of Lemma 8.1 also represent respectively the number of even (even terms) and odd (odd terms) steps of the initial succession $\{a\}_{n \varepsilon N}$ up to the term d_{id}.

8.4 Theorem on the relationship between orbit steps $\{a_n\}_{n \varepsilon N}$

Enunciation *Given a sequence $\{a_n\}_{n \varepsilon N}$ generated by any positive odd integer d_0 between the number of its even steps ip and the number of its odd steps id up to the generic odd term d_{id} the relation ip =* $\left\lceil log_2\left(3^{id} * \frac{d_0}{d_{id}}\right)\right\rceil$.

Demonstration It has already been pointed out (Remark 8.3) that id and ip, which appear in (8.1) and which are relative to the succession $\{d_n\}_{n \varepsilon N}$correspond to the odd-numbered id and even-numbered ip steps of the sequence $\{a_n\}_{n \varepsilon N}$ of which the $\{d_n\}_{n \varepsilon N}$ is almost equivalent. Let us then see on the basis of (8.2) what relation can be found between the id and the ip of the $\{d_n\}_{n \varepsilon N}$.

First of all, we can derive the following inequalities:

(8.5) $2^{ip} * d_{id} > 3^{id} * d_0$

and subtracting from both members the term $2^{ip-1} * d_{id}$ and knowing that 2^{ip}- $2^{ip-1} = 2^{ip-1}$

(8.6) $2^{ip-1} * d_{id} > 3^{id} * d_0 - 2^{ip-1} * d_{id} \Longrightarrow 2^{ip-1} * d_{id} < 3^{id} * d_0$

and so we can write:

(8.7) $2^{ip-1} * d_{id} < 3^{id} * d_0 < 2^{ip} * d_{id} \Longrightarrow 2^{ip-1} < 3^{id} * \frac{d_0}{d_{id}}$

From (8.5) we derive that:

(8.8) $2^{ip} > 3^{id} * \frac{d_0}{d_{id}}$

and also being $2^{ip-1} < 3^{id} * \frac{d_0}{d_{id}}$ it results that for each orbit of d_0 up to any d_{id} the number ip of even steps is related to the number id of odd steps by the following relation:

(8.9) ip = $\left\lceil log_2\left(3^{id} * \frac{d_0}{d_{id}}\right)\right\rceil$

9 One is the only d_{id} that repeats in the orbit $\{d_n\}_{n \varepsilon N}$

We now prove that in an orbit $\{d\}_{n \varepsilon N}$generated by any positive integer d_0 , no d_{id} of the orbit is equal to another d_{id} , starting from d_0, until the orbit reaches the number 1 and thus the orbit $\{a\}_{n \varepsilon N}$, of which $\{d\}_{n \varepsilon N}$ is almost equivalent, the cycle 4, 2, 1.

9.1 Theorem on the uniqueness of the 4, 2, 1 cycle in the orbit $\{a\}_{n \varepsilon N}$

Enunciated *In every Collatz orbit generated by any positive integer at_0 there exists only the infinite cycle 4, 2, 1*

Demonstration If we prove that no number repeats in an orbit $\{d\}_{n \, \varepsilon \, N}$ generated by any positive odd integer d_0 until the orbit reaches the number 1, which will instead repeat infinitely, we have also shown that there is no cycle in the orbit $\{a\}_{n \, \varepsilon \, N}$ until the orbit reaches the number 1. In fact, if no d_{id}, starting from d_0, repeats along the orbit $\{d\}_{n \, \varepsilon \, N}$ all even numbers $3d_{id}+1$ and those that result from successive divisions of the same by 2 will also not repeat in the corresponding orbit $\{a\}_{n \, \varepsilon \, N}$. On the other hand, infinitely in addition to 1, 4 and 2, which are equal to 3*1+1 and (3*1+1)/2, will repeat.

Let us now see why no term d_{id} of the succession repeats in an orbit $\{d\}_{n \, \varepsilon \, N}$ until it reaches the number 1.

We have already seen in (8.2) that every d_{id} belonging to the orbit $\{d\}_{n \, \varepsilon \, N}$ is equal to the ratio of two polynomials increasing, albeit differently, with the odd id steps of the orbit:

(9.2) $d_{id} = \frac{3^{id} * d_0 + 3^{id-1} + 3^{id-2} * 2^{s_1} + 3^{id-3} * 2^{s_1+s_2} + \cdots \dots\dots + 3 * 2^{s_1+\cdots+s_{id-2}} + 2^{s_1+\cdots+s_{id-1}}}{2^{s_1+\cdots+s_{id-1}} * 2^{s_{id}}}$

that by positing $\Delta = 3^{id-1} + 3^{id-2} * 2^{s_1} + 3^{id-3} * 2^{s_1+s_2} + \cdots \dots\dots + 3 * 2^{s_1+\cdots+s_{id-2}} + 2^{s_1+\cdots+s_{id-1}}$

and ip = $s_1 + s_2 + \cdots \dots\dots\dots + s_{id}$ becomes:

(9.3) $d_{id} = \frac{3^{id} * d_0 + \Delta}{2^{ip}}$

Thus, (9.3) expresses the relationship existing between any d_{id} of the orbit and its generator d_0 and where id is the index of the term d_{id} of the succession $\{d\}_{n \, \varepsilon \, N}$ (as well as the number of odd-numbered steps in the succession $\{a\}_{n \, \varepsilon \, N}$) and ip the sum of the exponents of the powers of 2 that divide the 3d+1 terms present between d_0 and d_{id} (as well as the number of even-numbered steps in the sequence $\{a\}_{n \, \varepsilon \, N}$).

Considering then any two nodes of the orbit $\{d\}_{n \, \varepsilon \, N}$ equal to d_{idx} and d_{idy}, respectively, with idy > idx, we can write:

(9.4) $d_{idx} = \frac{3^{idx} * d_0 + \Delta_x}{2^{ipx}}$ and $d_{idy} = \frac{3^{idy} * d_0 + \Delta_y}{2^{ipy}}$

where idx and idy are the indices of the terms d_{idx} and d_{idy} of the sequence $\{d\}_{n \, \varepsilon \, N}$ and ipx and ipy are the sums of the exponents of the powers of 2 that divide the various 3d+1 nodes present between d_{idx} and d_0 and between d_{idy} and d_0 respectively. It should be noted that since the term d_{idy} follows that of d_{idx} and since the numerators and denominators of (9.2) always increase as the nodes of the orbit $\{d\}_{n \, \varepsilon \, N}$ (regardless of the value of their non-monotonic ratio) we have that idy and ipy are always greater than idx and ipx respectively[1] .

If, absurdly, we assume that $d_{idy} = d_{idx}$, we should have accordingly:

(9.5) $\frac{3^{idy} * d_0 + \Delta_y}{2^{ipy}} = \frac{3^{idx} * d_0 + \Delta_x}{2^{ipx}}$

Now being for (8-7) $2^{ip-1} * d_{id} < 3^{id} * d_0$ and for (9.3) $2^{ip} * d_{id} - \Delta = 3^{id} * d_0$ we can write:

(9.6) $2^{ip-1} * d_{id} < 2^{ip} * d_{id} - \Delta$

[1] Also in the case that d_{idy} e d_{idx} are consecutive, it will be the case that idy=idx+1 and ipy will also be at least one unit greater than ipx because between the two odd terms d_{idx} and d_{idy} there is at least the even node corresponding to $3^*.d_{idx} + 1$

and being $2^{ip} - 2^{ip-1} = 2^{ip-1}$ we can deduce that $\Delta < 2^{ip-1} * d_{id}$ and therefore also that $\Delta < 3^{id} * d_0$ and that it is therefore finally possible to neglect Δ with respect to $3^{id} * d_0$.

This allows us to write the equality of (9.5) in good approximation as follows:

$$(9.7)\quad \frac{3^{idy} * d_0}{2^{ipy}} \approx \frac{3^{idx} * d_0}{2^{ipx}} \quad \Longrightarrow \quad 3^{idy-idx} \approx 2^{ipy-ipx}$$

equality the latter also follows from (8-9) by subtracting ipx from ipy and placing the difference as an exponent of the power of two:

$$(9.8)\quad 2^{ipy-ipx} \approx \frac{d_{idx}}{d_{idy}} * 3^{idy-idx} \quad \Longrightarrow \quad 3^{idy-idx} \approx 2^{ipy-ipx}$$

according to the initial hypothesis $d_{idy} = d_{idx}$. But this equality turns out to be impossible unless the two exponents of (9.7) are equal to zero, that is idy=idx and ipy=ipx, condition that does not exist for two distinct nodes of the orbit. The approximation of (9.8) does not invalidate the reasoning since the difference between the two powers grows on average very rapidly as the exponent of 3 increases, deriving from 8.9 that of 2.

Consequently, the hypothesis $d_y = d_x$ is false and the theorem proved.

Remark 9.9 We would arrive at the same formula (9.7) if we calculated idy and idx from formula (5.4) which for each node K_i of the orbit gives the approximate relationship between the output d_i and the input d_{i-1} :

(9.9) $d_i = (3d_{i-1} + 1)/2^{k_i} \approx 3d\,/_{i-1}2^{k_i}$

In fact, by repeating this formula, starting with d_0 , for the id nodes of the orbit $\{d\}_{n\,\varepsilon\,N}$, we will obtain:

(9.10) $d_{id} \approx 3^{id} * d_0/2^{ip}$

where id is the index of node k_{id} of the sequence $\{d\}_{n\,\varepsilon\,N}$ and ip is the sum of the various k_i of (9.9) between d_0 and d_{id} . From here on, considering any two nodes of the orbit $\{d\}_{n\,\varepsilon\,N}$ equal respectively to d_{idx} and d_{idy} , with idy > idx, the treatment is analogous to that carried out previously from (9.7) onwards.

10 Validity of the Collatz conjecture

Having proved (4) that the succession $\{a\}_{n\,\varepsilon\,N}$ always arrives at 1 if the equivalent succession also arrives at 1 $\{d\}_{n\,\varepsilon\,N}$ and that the latter always reaches 1 since:

a) the succession is superiorly limited (7) and therefore there is no odd number d for which the succession tends to infinity

b) up to the node of the orbit equal to 1 there are no numbers in it that repeat without ever giving 1 (9) and therefore no term of the succession $\{d_n\}_{n\,\varepsilon\,N}$ is equal to one of its preceding terms

the Collatz conjecture is proved

ANNEXO A

Number of consecutive times that the term 3di+1, starting with the initial term going through the INC process, is only divisible by 2
h=number of times; d=density calculated from the odd numbers in the set D1

of Ɛ D1	h=1 d1=(3di+1)/2; d=1/2	h=2 d2=(3d1+1)/2; d=1/4	h=3 d3=(3d2+1)/2; d=1/8	h=4 d4=(3d3+1)/2; d=1/16	h=5 d5=(3d4+1)/2; d=1/32	h=6 d6=(3d5+1)/2 d=1/64
3	5					
7	11	17				
11	17					
15	23	35	53			
19	29					
23	35	53				
27	41					
31	47	71	107	161		
35	53					
39	59	89				
43	65					
47	71	107	161			
51	77					
55	83	125				
59	89					
63	95	143	215	323	485	
67	101					
71	107	161				
75	113					
79	119	179	269			
83	125					
87	131	197				
91	137					
95	143	215	323	485		
99	149					
103	155	233				
107	161					
111	167	251	377			
115	173					
119	179	269				
123	185					
127	191	287	431	647	971	1457

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