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INVESTIGATING FERMAT'S LAST THEOREM WITH AN EXTENSION OF THE PYTHAGOREAN TERNS

Abstract

This article provides novel insights on Fermat's last theorem based primarily on an extension of the Pythagorean triples and their diagrams. The study arrives at a trivial proof of Fermat's last theorem and opens up possible new fields of research in the field of algebraic equations.

Foreword

Fermat's Last Theorem, or more correctly Fermat's Last Conjecture, states that there are no positive integer solutions of the equation:

$x^n + y^n = z^n$

if n > 2 and if xyz <> 0.

Fermat formulated the conjecture in 1637, but did not make known the proof he claimed to have found. Only in 1994 was the conjecture proved by Andrew Wiles through the use of elements of modern mathematics and algebra that Fermat could not have known at that time; therefore Fermat's proof, if it existed and was correct, could only be different.

The Diagram of the Pythagorean Triads

The equation of Fermat's last theorem can be considered an extension of the Pythagorean terns

$x^2 + y^2 = z^2$

which as we know are infinite to the exclusion of trivial solutions.

It is from studying these equations and extending them to exponents greater than 2 that the Pythagorean Triad Diagram was identified as a useful tool. It consists of a Cartesian plane in which on the x-axis are the positive integers k and to each column of its vertical grid is associated an odd positive integer starting from 1 (fig. 1).

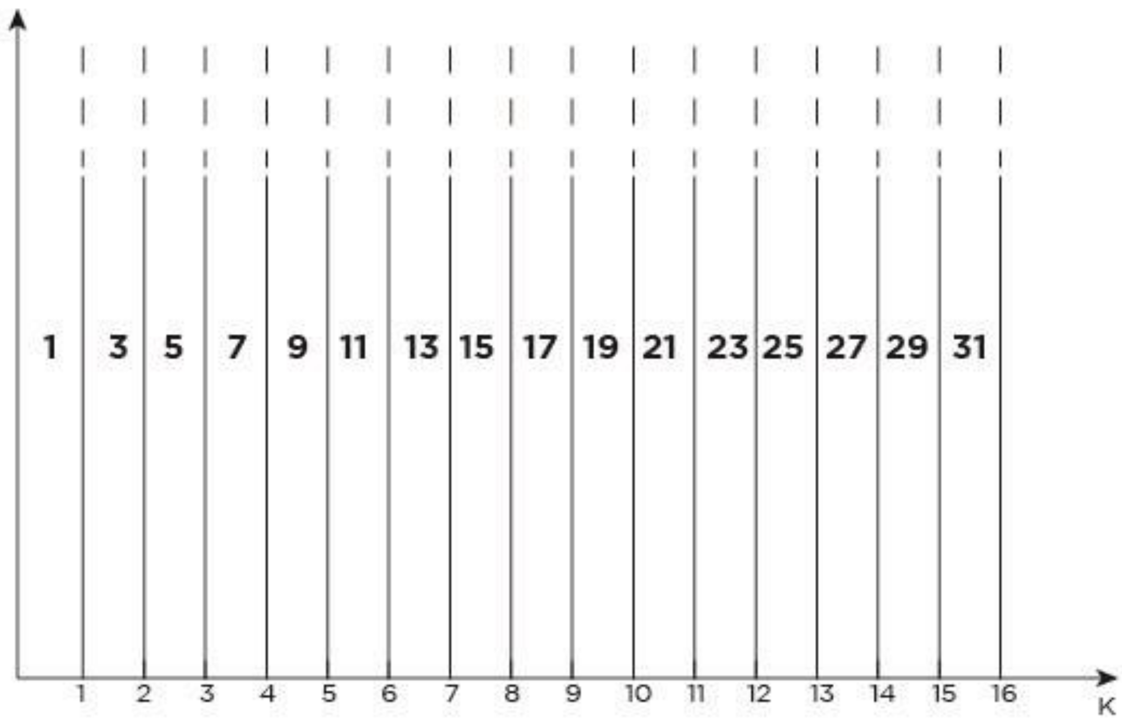

fig. 1 - Diagram of Pythagorean terns

From the analysis of the diagram in the figure, we can deduce that:

(a) the value of each column between k and k'=k+1 is equal to 2k+1, 2k'-1 and k+k'
(b) the sum of the values of all columns between 0 and k is equal to k^2
c) Consequently, the sum of the values of the columns between k_1 and k_2 ($\sum_2 - \sum_1$), with $k_2 > k_1$, is equal to $k_2^2 - k_1^2$ (see Fig. 2).

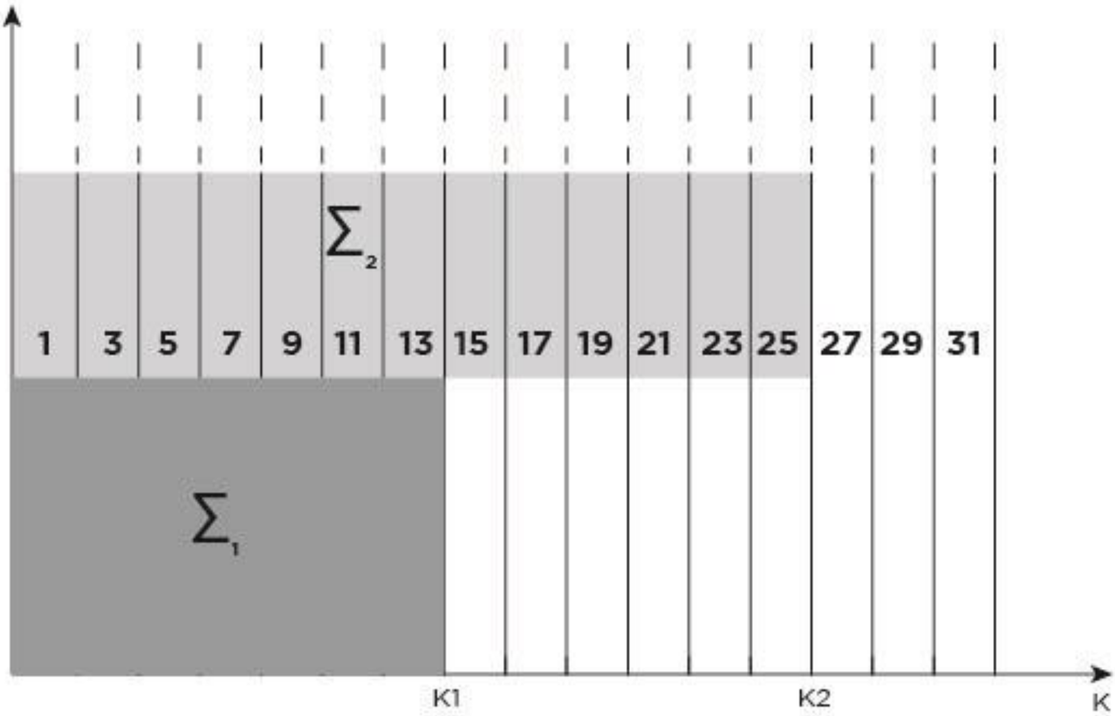

fig. 2 - $\sum_2 - \sum_1 = k_2^2 - k_1^2$

d) The sum of the values of an odd emmuple (with m>1) of columns always corresponds to an odd compound number:

In fact, if the emmuple is centred on the column of value 1+2k, the sum of its values will assume the expression:

(1 + 2k) + [(1 + 2k) -2] + [(1 + 2k) + 2] +
[(1 + 2k) -4] + [(1 + 2k) + 4] +
[(1 + 2k) -6] + [(1 + 2k) + 6] +
.... +
[(1 + 2k) -((m-1)/2)*2] + [(1 + 2k) + ((m-1)/2)*2] =

(1 + 2k) + (m-1)*(1 + 2k) = m*(1 + 2k)

where n = m*(1 + 2k) is an odd non-prime number for which there are as many different ennuples whose sum of values is equal to n, as there are all the possible products equal to n of two distinct natural numbers [m and (1+2k)]. Since m is any odd number and k is any positive integer, it follows that for every positive odd integer n there will always exist one or more ennuples whose sums of values are equal to n, and thus one or more pairs of values k_2 , k_1 such that $n = k_2^2 - k_1^2$. For each prime number p, on the other hand, there is no ennuple (with m>1) whose sum of values is equal to p, but there is always only one column (between k_2 and k_1) whose value is equal to p, and so there are again two values k , k_{21} such that $p = k_2^2 - k_1^2$ where $k_2 = (p+1)/2$ and $k_1 = (p-1)/2$.

e) The sum of the values of an even emmuple of columns always corresponds to an even number consisting of 4:

In fact, if the emmuple is centred on the two columns of value 2k-1 and 2k+1, the sum of its values will take on the expression

$$
\begin{aligned}
4k &+ [(2k - 1) -2] + [(2k + 1) + 2] + \\
&[(2k - 1) -4] + [(2k + 1) + 4] + \\
&[(2k - 1) -6] + [(2k + 1) + 6] + \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \\
&[(2k - 1) - ((m-2)/2)*2] + [(2k + 1) + ((m-2)/2)*2] =
\end{aligned}
$$

$$4k + ((m-2)/2)(2k-1) + ((m-2)/2)(2k+1) = 4k + (m-2)2k=2km$$

and, since m is even, the sum will be of the type 4km' with m'=m/2. Consequently, only the numbers n even positive integers multiples of 4 have one or more different ennuples whose sums of values are equal to n and thus one or more pairs of values k_2 , k_1 such that $n = k_2^2 - k_1^2$.

On the basis of the above we can state that to the term n^2 , this being either a compound odd number or an even number multiple of 4, it is always possible to associate one or more pairs of values k_2 , k_1 such that $n^2 = k_2^2 - k_1^2$. Assuming therefore that n=b, $k_2 = c$ and $k_1 = a$ for each positive integer b there exist one or more pairs of positive integers a and c such that it can be written:

$$c^2 = a^2 + b^2$$

Ultimately, it can be stated that for every natural number b there are one or more Pythagorean triples with the same integer b.

Extension of the Pythagorean terns

If in the Diagram of Pythagorean Triads, fig. 3, we select an emmuple between 0 and h and a subsequent value l (l > h) and translate the emmuple from 0 to l so that it occupies the columns between l and l+h, the sum of the values of the columns of the translated emmuple will be given by

1) $C_{(h,l)} = h^2 + 2lh$

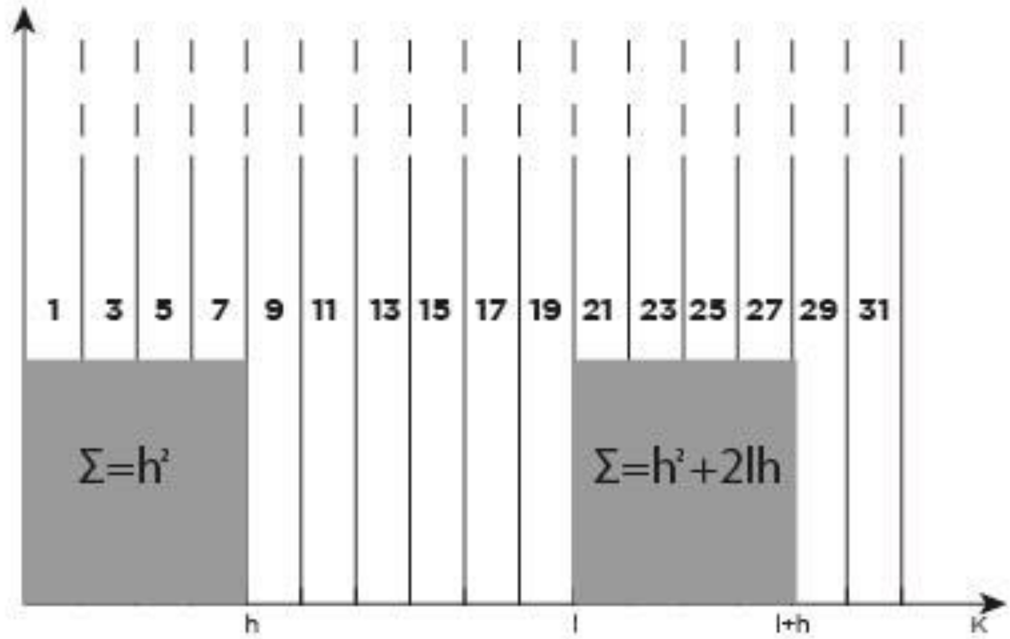

Fig. 3 - translation of an emmuple from 0 to l

As l varies, if h is odd, $C_{(h,l)}$ gives us all the odd compound numbers of h greater than its square and thus also, $\forall$ n $\in$ N and n greater than 2, the power h^n .
If, on the other hand, h is even $C_{(h,l)}$ gives us all the even compound numbers, multiples of 4 of h and greater than its square and thus also, $\forall$ n $\in$ N and n greater than 2, the power h^n , which, being h even and n>2, will surely be a multiple of 4.

We can then deduce that for each $Y = y^n$ with y and n belonging to N and with natural n greater than 2, there are as many different ennuples, whose sum of values is equal to Y, as there are possible products equal to Y of two distinct natural numbers (factors or products of factors of Y), and in particular of y and y^{n-1} . Consequently, the following equalities will occur:

2) $Y = C_{(y,ly)} = y^2 + 2l\ y_y$

where l_y represents the initial value l of the translated ennuple (0,y); and

3) $Y = C_{(f,lf)} = f^2 + 2l\ f_f$

where f and l_f represent respectively a factor (or product of factors) of Y and the initial value l_f of the translated ennuple (0,f).

Fermat's Conjecture

The conjecture, as already written, states that there are no positive integer solutions of the equation:

4) $x^n + y^n = z^n$

if n > 2 and if xyz <> 0

or, equivalently, that there are no positive integer solutions of the equation:

5) $z^n - x^n = y^n$

always if n > 2 and if xyz <> 0

Let us assume by hypothesis that the conjecture is false for *even n,* i.e. that *for even n greater than 2* there exists at least one triplet of positive integer values x, y, z such that the equation $z^n - x^n = y^n$ has at least one solution.

Now with n even integer greater than 2, $z^{n/2}$ and $x^{n/2}$ are positive integers, and for any pair z, x the sum of the values of the columns of the emmuple between $z^{n/2}$ and $x^{n/2}$ in the diagram of quadratic triads will either be an odd compound number (since it cannot be verified that $z^{n/2} - x^{n/2}$ is equal to 1) or an even multiple of 4, which we shall denote by Y. That is, we will have $Y = (z)^{n/22} - (x)^{n/22} = z^n - x^n$.

If, according to the hypothesis, $Y=y^n$ we will have that $Y=y*y^{n-1}$ is a compound of y and, according to the law of compound numbers in the diagram of quadratic triads (whereby any compound of y greater than its square y^2 will be given by the sum of y columnar values relative to a sequence of origin l > 0) we can write: $y = z^{n/2} - x^{n/2}$.

Then, being by construction $y = z^{n/2} - x$ and $Y^{n/2} = z^n - x^n$, and being by hypothesis $Y = y^n$ we have:

$z^n - x^n = (z^{n/2} - x)^{n/2n}$ and thus:

$(z^{n/2} - x^{n/2})*(z^{n/2} + x^{n/2}) = (z^{n/2} - x^{n/2})*(z^{n/2} - x)^{n/2n-1}$

and dividing both members by $(z^{n/2} - x)^{n/2}$ has:

6) $(z^{n/2} + x^{n/2}) = (z^{n/2} - x)^{n/2n-1}$

We now show that 6), for any pair of values $z^{n/2}$ $x^{n/2}$ (with n > 2 and xz <> 0), is incorrect because it always results as we shall see:

7) $(z^{n/2} + x^{n/2}) < (z^{n/2} - x)^{n/2n-1}$

Let us begin by saying that with n *even* > 2, n-1 will never be less than 2 and will always result $| z^{n/2} - x^{n/2} | > 2$.

That being said:

$(z^{n/2} - x)^{n/2n-1} = (z^{n/2} - x)^{n/22} *(z^{n/2} - x)^{n/2n-3}$

7) will always be true since, under the given conditions of n *even* > 2 and $| z^{n/2} - x^{n/2} | > 2$, the inequality is valid:

8) $(z^{n/2} + x^{n/2}) < (z^{n/2} - x)^{n/22}$

It can easily be verified that by placing the two members of 8) in the condition of least difference, i.e. n=4 and z=x+1, the result is:

$$(z^{n/2} - x)^{n/22} = 2*(z^{n/2} + x^{n/2}) - 1$$

In conclusion, we can then say that the initial hypothesis $z^n - x^n = y^n$ proves to be false and it therefore follows that Fermat's conjecture *for n even positive integer* is true.

Let us now assume instead by hypothesis that the conjecture is false for *n odd,* i.e. that *for n odd and greater than 2* there exists at least one triplet of positive integer values x, y, z such that the equation $z^n - x^n = y^n$ has at least one solution.

But if Fermat's conjecture is true for an exponent 'n' , it is also true for every multiple of n. In fact, for every k > 1, we have that:

$$z^{kn} - x^{kn} = y^{kn} \Longleftrightarrow Z^n - X^n = Y^n \text{ where } Z = z^k , X = x^k , Y = y^k$$

Thus, if there were a triplet of positive integer values x, y, z such that the equation $z^d - x^d = y^d$ *with d an odd positive integer* would be true, then every equation $z^{kd} - x^{kd} = y^{kd}$ *with k even would* have to be true, which, *since kd is obviously an even positive integer*, contradicts what was shown earlier about the veracity of Fermat's conjecture *for every n even.*

In conclusion, Fermat's conjecture is proven for both n, integers, positive and greater than 2, even and odd.