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Congruence Primality Theorems and Complementary Congruence and Density of Incongruent Numbers

Abstract

In the article, a study of congruence and complementary congruence is developed that arrives at the definition of two primality theorems of congruence and compcongruence and the density function of incongruent numbers. The study is preparatory to the demonstration of the Hardy-Littlewood and Goldbach conjectures. The study and opens up new areas of possible research in the field of Number Theory..

1 Congruence Primalities

1.1 The Congruence of Natural Numbers

As is well known, the congruence relation [1.2.1 of (a)] modulus m is an equivalence relation defined on the set of integers Z as follows: if m is a fixed integer greater than 1, two integers a and b are said to be congruent modulus m if m|(a - b); m is called the modulus of congruence and is denoted by a ≡ b (mod m).
In the field of natural numbers, it can also be equivalently stated that a ≡ b (mod m) if a and b give the same remainder in the integer division by m.
For example, 24 ≡ 10 (mod 7) because they both give remainder 3 in the integer division by 7. All numbers congruent with each other modulo m constitute an equivalence class, called the congruence class modulo m: two natural numbers belong to the same congruence class if and only if they are congruent modulo m, that is, if and only if they divide by m and give the same remainder r. If, as in the example, the modulus is 7, seven classes are thus formed (as many as there are possible remainders in the division by 7) as follows [0], [1], [2], [3], [4], [5], [6]. Always limiting ourselves to the subset of Z consisting of the natural numbers, to establish to which class modulo m one of them belongs we divide it by m, the remainder indicating the class.
It should be emphasised that for each m it is always the case that $[m]_{\text{mod m}} = [0]_{\text{mod m}}$.

Remark 1.1.2 From Number Theory we know that any natural number n will only be non-prime if it is divisible by one or more prime numbers less than or equal to the $\sqrt{n}$. Since all even natural numbers, except 2, are non-prime because they are divisible by 2, it can be asserted that any odd natural number n > 4 will only be non-prime if it is divisible by one or more prime numbers odd less than or equal to . $\sqrt{n}$.

From here on, the variables p, p_1, p_2, . . . p_i always denote prime numbers and $\mathbb{P}(M)$ the set of odd prime numbers less than or equal to the number M.

1.2 Congruence Primality Theorem

Enunciation 1.2.1 *∀ N_0 , n_0∈ N with $N_0 \geq 3$, $0 \leq n_0 \leq N_0 - 3$ and even if N_0 is odd or vice versa, with $\mathbb{P}\left(\sqrt{(N_0 - n_0)}\right)$ set of odd prime numbers ≤ $\sqrt{(N_0 - n_0)}$, a necessary and sufficient condition for $N_0 - n_0$ to be a prime number is that $n_0 \not\equiv N_0 \pmod{p_i}$ ∀ $p_i \in \mathbb{P}\left(\sqrt{(N_0 - n_0)}\right)$ or that $\mathbb{P}\left(\sqrt{(N_0 - n_0)}\right)$ is an empty set.*

Dim. According to the congruence of natural numbers (1.1) if N_0 and n_0 do not belong to the same congruence class modulo p_i for all $p_i \in \mathbb{P}\left(\sqrt{(N_0 - n_0)}\right)$, this means that $N_0 - n_0$ (an always odd natural number) is not divisible by any odd prime number less than or equal to the $\sqrt{(N_0 - n_0)}$ and that therefore, according to observation (1.1.2), $N_0 - n_0$ is a prime number. If instead $\mathbb{P}\left(\sqrt{(N_0 - n_0)}\right)$

results in an empty set (with $n_0 = N_0 - 3, N_0 - 4, N_0 - 5, N_0 - 6, N_0 - 7, N_0 - 8$) the number $N_0 - n_0$ cannot be divided by any prime and is therefore prime.
Conversely, if $N_0 - n_0$ is a prime number, it will not be divisible by any other lower, equal or non-existent odd prime number of the $\sqrt{(N_0 - n_0)}$ and therefore N_0 and n_0 will always result non congrui $\forall p_i \in \mathbb{P}(\sqrt{(N_0 - n_0)})$.

We set $n_0 \leq N_0 - 3$ because with $n_0 = N_0 - 1$ one would have that $N_0 - n_0 = 1$ which, as is known, is neither a prime nor a compound number, and with $n_0 = N_0 - 2$ one would have that N_0 and n_0 would both be even or odd contrary to the hypothesis. In order then to prevent n_0 from taking negative values, it must be $N_0 \geq 3$.

Remark 1.2.2 *If, instead of referring to the set* $\mathbb{P}(\sqrt{(\boldsymbol{N_0} - \boldsymbol{n_0})})$ *we want to refer, for the needs of successive demonstrations, to the set* $\mathbb{P}(\sqrt{\boldsymbol{N_0}})$*, the theorem (1.2.1) is transformed into the corollary (1.2.3)*

Given a number $N_0 \in N$, a number $n_0 \in N$, smaller than N_0 and such that $(N - n_{00})$ is odd is called the **Prisotto of N_0** if it turns out that $n_0 \not\equiv N_0 \pmod{p_i}$ $\forall p_i \in \mathbb{P}(\sqrt{(N_0)})$.

Corollary 1.2.3 *$\forall N_0, n_0 \in N$ with $N_0 \geq 9$, $0 \leq n_0 \leq N_0 - p_{max}$ and even if N_0 is odd or vice versa, with $\mathbb{P}(\sqrt{(N_0)})$ set of odd prime numbers $\leq \sqrt{(N_0)}$ and with p_{max} prime number higher than $\mathbb{P}(\sqrt{(N_0)})$, a necessary and sufficient condition for $N_0 - n_0$ to be a prime number is that n_0 is a prime number of N_0.*

Dim. substituting $\mathbb{P}(\sqrt{(N_0)})$ a $\mathbb{P}(\sqrt{(N_0 - n_0)})$, in contrast to theorem (1.2.1), the numbers n_0 smaller than N_0 and belonging to the interval $[N - p_{0max}., N_0 - 3]$ are not considered since they all have at least one congruence class mod p_j, with $p_j \in \mathbb{P}(\sqrt{(N_0)})$, equal to that of the same modulus of N_0. In fact for the $n_0 \in [N_0 - p_{max}., N_0 - 3]$, $N_0 - n_0$ will belong to the interval $[3, p_{max}]$ and thus be equal to a prime or compound number belonging to this interval; in the first case according to modular arithmetic if $N_0 - n_0 = p_j$, with $p_j \in \mathbb{P}(\sqrt{(N_0)}) \subset [3, p_{max}.]$ this implies that $[N_0] \bmod p_j - [n] \bmod p_{0j} = [p_j] \bmod p_j = [0]$ whence the congruence mod p_j of n_0 with N_0; if instead $N_0 - n_0$ is equal to a compound number $m^* p_j$, with $p_j \in \mathbb{P}(\sqrt{(N_0)}) \subset [3, p_{max}.]$, we will have that $[N_0] \bmod p_j - [n_0] \bmod p_j = [m] \bmod p_j * [p_j] \bmod p_j = [m] \bmod p_j * [0] = [0]$ whence the congruence mod p_j of n_0 with N_0.
Conversely, if $N_0 - n_0$ is a prime number, belonging to the interval $]p_{max}, N_0]$, it as prime will not be divisible by any other odd prime number less than or equal to p_{max} and thus the $\sqrt{(N_0)}$ and therefore N_0 and n_0 will always be non congrui $\forall p_i \in \mathbb{P}(\sqrt{(N_0)})$.

He placed himself $N_0 \geq 9$ in quanto per valori inferiori p_{max} would not be defined.

According to Corollary 1.2.3, we can state that the numbers n_0 prisotto of N_0, subtracted from N_0, result in all prime numbers in the interval $]p_{max}, N_0]$.

Remark 1.2.4 *Obviously, with N_0, being equal, the difference between the set of incongruous numbers less than N_0 modulo $\mathbb{P}(\sqrt{(\boldsymbol{N_0} - \boldsymbol{n_0})})$ and that of the numbers less than N_0 is given by all $n_0 = N - p_{0i}$ with $p_j \in \mathbb{P}(\sqrt{(\boldsymbol{N_0})})$. In practice, the number of all odd primes less than or equal to N_0 is equal to the sum of the number of prisotto numbers of N_0 and that of $p_j \in \mathbb{P}(\sqrt{(\boldsymbol{N_0})})$.*

Remark 1.2.5 *Both theorem (1.2.1) and corollary (1.2.3) tell us nothing about the existence of at least one incongruous n_0 . However, according to postulate [6.3 of (b)] of Bertrand (later proved by Pafnuty Chebyshev, Srinivasa Ramanujan and Paul Erdős), which states that for each $n \geq 2$ there exists at least one prime p such that $n < p < 2n$, we can state, with respect to the corollary (1.2.3), that in the interval] p_{max} , N_0] there will always exist at least one prime being $2\,p_{max} \leq 2\sqrt{N_0} \leq N_0$ for $N_0 \geq 4$. Consequently, in the interval]0, N_0 - p_{max} [there will always exist at least one n_0 prisot of N_0.*

1.3 The Compcongruence of Natural Numbers

We now introduce **Complementary Congruence** (compcongruence) modulus m as the correspondence relation defined on the set of integers Z as follows: if m is a fixed integer greater than 1, two integers a and b are said to be compcongruent modulus m if m|(a + b); m is called the modulus of the compcongruence and we will denote it by a ‖ b (mod m).

In the field of natural numbers, one can also equivalently state that a ‖ b (mod m) if a and b give two complementary remainders with respect to m in the integer division by m. For example, 24 ‖ 39 (mod 7) because they give as remainders in the integer division by 7 respectively 3 and 4, i.e. two complementary numbers with respect to 7.

1.4 Compcongruence Primality Theorem

Enunciation 1.4.1 *$\forall N_0$, $n_0 \in N$ with $N_0 \geq 2$, $0 \leq n_0 \leq N_0 - 1$ and even if N_0 is odd or vice versa, with $\mathbb{P}\left(\sqrt{(N_0+n_0)}\right)$ set of odd prime numbers $\leq \sqrt{(N_0+n_0)}$, a necessary and sufficient condition for $N_0 + n_0$ to be a prime number is that $n_0 \nparallel N_0$ (mod p_i) $\forall p_i \in \mathbb{P}\left(\sqrt{(N_0+n_0)}\right)$ or that $\mathbb{P}\left(\sqrt{(N_0+n_0)}\right)$ is an empty set.*

Dim. According to the compcongruence of natural numbers (1.3), if N_0 and n_0 are **not** compcongruent modulo p_i for all p_i belonging to the ***set*** $\mathbb{P}\left(\sqrt{(\boldsymbol{N_0}+\boldsymbol{n_0})}\right)$, this means that $N_0 + n_0$ is not divisible by any prime number less than the $\sqrt{(\boldsymbol{N_0}+\boldsymbol{n_0})}$ and that therefore, according to observation (1.1.2), $N_0 + n_0$ is a prime number. If, on the other hand $\mathbb{P}\left(\sqrt{(\boldsymbol{N_0}+\boldsymbol{n_0})}\right)$ is an empty set, the number $N_0 + n_0$ cannot be divided by any prime and is therefore prime.
Conversely, if $N_0 + n_0$ is a prime number, it will not be divisible by any other lower, equal or non-existent odd prime number of the $\sqrt{(N_0+n_0)}$ and therefore N_0 and n_0 will always be non compcongrui $\forall p_i \in \mathbb{P}\left(\sqrt{(N_0+n_0)}\right)$.

We set $N_0 \geq 2$ because with N_0 =1 and n_0 =0 we would have N +n_{00} =1, a non-prime and non-compound number.

Remark 1.4.2 *If, instead of referring to the set $\mathbb{P}\left(\sqrt{(\boldsymbol{N_0}+\boldsymbol{n_0})}\right)$ we want to refer, for the needs of successive demonstrations, to the set $\mathbb{P}\left(\sqrt{\boldsymbol{2N_0}}\right)$ the theorem (1.4.1) is transformed into the corollary (1.4.3).*

That is, given two numbers N_0 , $n_0 \in N$, with $n_0 < N_0$ and such that (N_0 +n_0) is odd, if it turns out that every odd prime number $p \leq \sqrt{(2N_0)}$ does not divide the number (N_0 +n_0) it means that it is prime.

Given a number $N_0 \in N$, a number $n_0 \in N$, less than or equal to N_0 and such that (N +n_{00}) is odd is called the **Prisopra of N_0** , if it turns out that $n_0 \nparallel N_0$ (mod p_i) $\forall\, p_i \in \mathbb{P}\left(\sqrt{(2N_0)}\right)$.

Corollary 1.4.3 *∀ N_0, n_0∈ N with $N_0 \geq 2$, $0 \leq n_0 \leq N_0 -1$ and even if N_0 is odd or vice versa, with $\mathbb{P}(\sqrt{(2N_0)})$ set of odd prime numbers ≤ $\sqrt{(2N_0)}$, a necessary and sufficient condition for $N_0 + n_0$ to be a prime number is that n_0 is a prisum of N_0 .*

Dim. Extending the set of prime numbers of theorem (1.4.1) from $\mathbb{P}(\sqrt{(\boldsymbol{N_0}+\boldsymbol{n_0})})$ **a** $\mathbb{P}(\sqrt{(\boldsymbol{2N_0})})$ and indicating by $\mathbb{P}(\Delta 2N_0)$ the set of primes in $\mathbb{P}(\sqrt{(2N_0)})$ e non in $\mathbb{P}(\sqrt{(N_0+n_0)})$nothing changes since for each of the numbers n_0 (incompcongrui con N_0 moduli $\mathbb{P}(\sqrt{(N_0+n_0)})$ tali che N_0+n_0= p_j, with p_j belonging to the interval [N_0 ,2N_0], it can never be the case that n_0 is compcongruent with N_0 modules $\mathbb{P}(\Delta 2N_0)$, e cioè che $[N_0]_{mod\ p_i}$+ $[n_0]_{mod\ p_i}$= $[0]_{mod\ p_i}$, for at least one $p_i \in$ $\mathbb{P}(\Delta 2N_0)$ **.** In fact, bearing in mind that $\sqrt{2N_0} \leq N_0$ con $N_0 \geq 2$ and that therefore all the primes p_i belonging to the set $\mathbb{P}(\Delta 2N_0)$ are ≤ N_0 we have that for each p_j belonging to the interval [N_0 ,2N_0] results $[p_j]_{mod\ p_i} \neq [0]$ always being p_i *e* p_j two prime numbers and different from each other. Consequently for each of the numbers n_0 tali che N_0+n_0= p_j, since modular arithmetic always results in [N_0] mod p_j + [n_0] mod p_j = $[p_j]_{mod\ p_i}$ and the latter is always different from zero, it can be stated that n_0 is prisopers of N_0 .
Conversely, if N_0 + n_0 is a prime number, it will not be divisible by any other lower, equal or non-existent odd prime number of the $\sqrt{(2N_0)}$ and therefore N_0 and n_0 will always result non compcongrui $\forall p_i \in \mathbb{P}(\sqrt{(2N_0)})$.

Remark 1.4.4 *Both the theorem (1.4.1) and the corollary (1.4.3) tell us nothing about the existence of at least one n_0 prisopra of N_0 . But on the basis of Bertrand's postulate [6.3 of (b)] we can state that in the interval [N_0 , 2N_0] there will always exist at least one prime and consequently in the interval]0, N_0] there will always exist at least one n_0 prisopra of N_0.*

1.5 Numbers and their congruence classes

Number Theory tells us that just as there exists in positional number systems (e.g. the decimal system) a bi-univocal correspondence between all possible numbers expressible with n digits (and therefore belonging to the interval]0, 10^n -1]) and all possible combinations (10^n) of the 10 digits, similarly there exists a bi-univocal correspondence between all possible numbers of the interval]0, p_{max}#], with p_{max} any prime and p_{max} # its prime, and the combinations of the congruence classes of these numbers having for modulus the single primes less than and equal to p_{max} . The existence of this biunivocal correspondence is easily proved by resorting to the Chinese Remainder Theorem [2.3.3 of (b)] and inserting as modules of the system of equations p_{max} and all primes less than it.

Remark 1.5.1 We will call the **table number-classes $\boldsymbol{p_{max}}$** the table which, for each number in the interval]0, p_{max}#] associates the combination of the congruence classes of this number having for modulus the single prime numbers less than and equal to p_{max} .

For illustrative purposes, let us consider (see Appendix A) a **number-class table** 7 containing for each number the corresponding combination of its 4 congruence classes mod 2, mod 3, mod 5 and mod 7.
The above-mentioned bi-univocal correspondence can be verified in this table. E.g. the combination 1-2-2-3 of the congruence classes mod 2, mod 3, mod 5 and mod 7 corresponds only to the number 17 in the interval [1, 210], just as the number 151 corresponds only to the combination 1-1-1-4 of the same congruence classes in the interval [1, 210].

As we shall see later, the Number-Class Table $\boldsymbol{p_{max}}$ is introduced in this study in order to calculate the densities of the numbers prisotto and prisopra of $\boldsymbol{N_0}$.

1.6 From Number-Class Table p_{max} to Primalities

Is there a criterion for deducing from the number-class table p_{max} and from the information contained therein how many, in addition to the modules {2,3,......, p_{max} } on which the table is built, are the prime numbers less than or equal to any $N_0 \in]0, p_{max}\#]$ and those within the interval [N_0 , 2 N_0] ?

Remark 1.6.1 *Among the various possible criteria, the one that interests us for our subsequent demonstrations consists in the application of the corollary (1.2.3) of the Primality of Congruence and that (1.4.3) of the Primality of Compongruence according to which the number of the odd primes less than or equal to N_0 is, minus the primes less than the $\sqrt{(\boldsymbol{N_0})}$ e cioè i modules {2,3,......, p_{max} } on which the table is constructed, is equal to that of the numbers in the table prisot of N_0 while the number of primes in the interval* [N_0 , 2 N_0] is equal to that of the numbers prisot of N_0.

From this observation, it follows that in order to derive from the number-class table p_{max} the prime numbers less than or equal to N_0 using the Primality of Congruence criterion, one must impose a condition that binds N_0 to the number-class table p_{max} and that is that the modules of the table must be exactly all the primes less than or equal to the $\sqrt{(N_0)}$.

In the case of our example table [1, 210] we can say that only for the N_0 such that $7 \leq \sqrt{(N_0)} < 11$, i.e. for N_0 greater than or equal to 49 and less than 121 we can say that the numbers in the table n_0 prisotto of N_0 are such that $N_0 - n_0$ is a prime number.

Similarly, to infer from the table-interval $]0, p_{max}\#]$ and from the information contained therein how many prime numbers there are in the interval [N_0 , $2N_0$] with $N_0 \in]0, p_{max}\#]$ using the criterion (Corollary 1.4.3) of the Primality of Compcongruence we must impose that the modules {2,3,......, p_{max} } of the table are exactly all primes less than or equal to the $\sqrt{(2N_0)}$. With this condition we will have that the numbers of the table incompcongruent less than N_0 are prime over N such that$_0$ such that $N_0 + n_0$ is a prime number.

In the case of our table [1, 210] for example we can say that only for the N_0 such that $7 \leq \sqrt{(2N_0)} < 11$ and i.e. for N_0 greater than or equal to 25 and less than 61 we can say that the numbers n_0 incompcongruous less than N_0 are prisopra of N_0 and i.e. that added to N_0 give the prime numbers of the interval [N_0 , $2N_0$].

1.7 From N_0 to the primes of the interval]0, $2N_0$]

If then, fixed at any $N_0 \in N$ greater than 49, we want to find out how many prime numbers are less than or equal to N_0 we must first find the highest prime number p_{max} less than or equal to $\sqrt{(N_0)}$ and then consider the number-class table p_{max} $]0, p_{max}\#]$, where $p_{max}\#$ is the prime of p_{max} and corresponds to the product of prime numbers $\leq p_{max}$ *Since the prime $p_{max}\#$ coincides with the prime $\sqrt{(\boldsymbol{N_0})}$ # in the remainder of the study we will write either $]0, p_{max}\#]$ o $]0, \sqrt{(\boldsymbol{N_0})}\ \#]$ to indicate the same* Number-class table p_{max} .

Remark 1.7.1 *The condition that N_0 is greater than or equal to 49 follows from the requirement that N_0 belongs to the interval $]0, p_{max}\#]$.*

From what is written in observation (1.6.1) the number of primes less than or equal to N_0 is given to us, minus the primes less than the $\sqrt{(\boldsymbol{N_0})}$ e cioè i forms 2, 3, ..., p_{max} on which the table is constructed, by that of the numbers in the table prism of N_0 , a number which, according to observation (1.2.5) will always be equal to or greater than 1.

E.g. with N_0 = 315 we will have that $\sqrt{315}$=17.746 and therefore p_{max} will be equal to 17, $p_{max}\#$ (2*3*5*7*11*13*17) will equal 510510 and the number 315 will correspond, in the interval [0 ,

p_{max}#], one and only one combination of its congruence classes mod 2, mod 3, mod 5, mod 7, mod 11, mod 13 and mod 17. All n_0 less than N_0 and incongruent with it with respect to $p_i \leq p_{max}$ i.e. all n_0 less than N_0, sottratti ad N_0 result in all prime numbers less than N_0 , except the primes 2,3,5,7,11,13,17 on which the table is built. On the other hand, according to the corollary (1.2.3) and observation (1.6.1), nothing can be said about the other numbers della tabella m_0 magggiori di 315 and incongruous with it p modules$_i$ belonging to $\mathbb{P}\left(\sqrt{(315)}\right)$.

Similarly, if we want to find, via a number table p_{max}, how many prime numbers there are in the interval [N_0 , $2N_0$] with any $N_0 \geq 121$, we must first find the highest prime number p_{max} less than the $\sqrt{2N_0}$ and then consider the number-class table p_{max}]0, $\sqrt{(2\boldsymbol{N_0})}$#].

Remark 1.7.2 *Here too, the condition that N_0 is greater than or equal to 121 follows from the requirement that $2N_0$ belongs to the interval]0, $\sqrt{(2\boldsymbol{N_0})}$#].*

According to observation (1.6.1), the number of primes present in the interval [N_0, $2N_0$] is given to us by that of the numbers in the table above N_0 , a number which according to observation (1.4.4) will always be equal to or greater than 1.

If we maintain the previous example of $N_0 = 315$, we must in this case calculate the $\sqrt{2 * 315}$ which is 25.1, from which it follows that p_{max} will be equal to 23, $\sqrt{(2\boldsymbol{N_0})}$# (equal to 2*3*5*7*11*13*17*19*23) will be equal to 223092870 and the number 315 will correspond, in the interval]0, $\sqrt{(2\boldsymbol{N_0})}$#], one and only one combination of its congruence classes mod 2, mod 3, mod 5, mod 7, mod 11, mod 13, mod 17, mod 19, mod 23. All n_0 less than N_0 and incompcongruent with it, i.e. all n_0 above N_0, sommati ad N_0 will result in all prime numbers in the interval [N_0 **,** **2** N_0]. On the other hand, according to the corollary (1.4.3) and observation (1.6.1), nothing can be said about the other numbers della tabella m_0 maggiori di 315 and incompcongruent with it modules $\mathbb{P}\left(\sqrt{(315)}\right)$.

2 The distribution of prime numbers

2.1 Theorem fundamental theorem of prime numbers

Gauss's Conjecture, dating back to 1792 and later becoming the Prime Number Theorem (NPT), on the distribution of prime numbers is:

(2.1.1) $\pi(N) \approx \frac{N}{logN} \approx \int_2^N \frac{dt}{logt} \approx Li(N)$

where π(N) is the number of primes less than or equal to N.

This conjecture was first proved in 1986 by Hadamard and de La Vallée Poussin using methods from the theory of complex functions related to the properties of Riemann's ζ-function. Mathematicians of the time, and in particular G. H. Hardy, believed that complex analysis was necessarily involved in the Theorem and that methods with only real variables were to be considered inadequate. But in 1949, Erdös and Selberg [3.4 of (a)] independently published an elementary proof (i.e. with only real variables), based on the combinatorial technique, of the Prime Number Theorem.

The demonstration of Selberg - Erdős [3.4 of (a)] thus brought into play the supposed superiority (depth) of complex analysis for the demonstration of NPT, showing that even technically elementary methods, which we have also adopted in this study, have their demonstrative effectiveness.

2.2 The average density of the n_0 incongruous of N_0 in the table]0, $\sqrt{(\boldsymbol{N_0})}$#]

Having fixed any $\boldsymbol{N_0} \in \boldsymbol{N}$ greater than 49, let us consider (see paras. 1.6 and 1.7) the relevant number-class table p_{max} of the interval]0, p_{max} #], where p_{max} is the highest prime number less than or equal to the $\sqrt{(N_0)}$, and calculate the number of all (greater than and less than N_0) the n_0 incongruous of N_0 present in the table.

We then eliminate from this table the rows that have one or more classes of congruence of the p modules$_i$ (2, 3, 5, ... , p_{max}) equal to the class corresponding to the remainder of N_0 for the same modules.

The numbers M in the table, not eliminated by the previous sieve, can then only be those which in the number-class table p_{max} have for each $p_i \in \mathbb{P}(\sqrt{(N_0)})$ one of the p_i -1 possible congruence classes other than the corresponding N_0 . (If e.g. (N_0) mod7 = 3, (M) mod7 must be equal to one of the 6 (7-1) other possible congruence classes: 0,1,2,4,5,6)

The rows of the table that have not been deleted will then, according to the combinatorial calculation, be:

(2.2.1) $\prod_{p=2}^{p_{max}}(p-1)$

Thus, (2.2.1) gives us the quantity of all **M** numbers in the table **incongruous (less than and greater than) than N_0 for the p modules only$_i$** belonging to the set $\mathbb{P}(\sqrt{(N_0)})$.

Let us now calculate the **average density Dnc**$_{]0, \sqrt{N_0}\#]}$ of these numbers M existing in the interval]0, $\sqrt{(N_0)}$ #] with $\sqrt{(N_0)}$ # = 2*3*.......*p_{max}can be written:

(2.2.2) $Dnc_{]0, \sqrt{N_0}\#]} = \frac{\prod_{p=2}^{p_{max}}(p-1)}{2*3*...*pmax} = \frac{\prod_{p=2}^{p_{max}}(p-1)}{\prod_{p=2}^{p_{max}}p} = \prod_{p=2}^{p_{max}}\frac{(p-1)}{p}$

[*formula this multiplied by* $\sqrt{(N_0)}$ *# corresponds to the Euler function φ(n) with n=*$\sqrt{(N_0)}$ *#, and gives the number of coprimes less than* $\sqrt{(N_0)}$ *#, a number that also includes the number of primes less than* N_0 *except for the primes belonging to the* set $\mathbb{P}(\sqrt{(N_0)})$]

On the basis of the corollary (1.2.3) of the Primality of Congruence and the fact that all numbers M less than N_0 (**M_{N0}**) are **prisot of N_0**, we can state that, for each of these numbers M_{N0} , N -M_{0N0} is a prime number and that the average density $Dnc_{]0,N_0]}$ of M_{N0} in the interval]0, N_0] is given by**:**

(2.2.3) $Dnc_{]0,N_0]} = \frac{Q(M_{N0})}{N_0}$ denoting by Q(M)$_{N0}$ the number of M_{N0} present in the interval]0, N_0].

As per observation (1.2.4) the number of **all** primes π(N)$_0$ less than or equal to N_0 is given by the sum of the number of M_{N0} and that of all $p_j \in \mathbb{P}(\sqrt{(N_0)})$ which, as we know, are not among the N_0 - M_{N0} .

We then know from NPT (2.1) that the average density $Dprimi_{N0}$ of the prime numbers less than N_0 , which coincides, barring the p_i belonging to the ***set*** $\mathbb{P}(\sqrt{(N_0)})$, with the average density Dnc_{N0} of the numbers M_{N0} **prisotto of N_0** is given by:

(2.2.4) $Dprimi_{]0,N_0]} = \frac{\pi(N_0)}{N_0} = \frac{1}{logN_0} \approx Dnc_{]0,N_0]}$

whence:

(2.2.5) $M_{N0} \approx \pi(N_0) \approx \frac{N_0}{logN_0}$

That is, for the density $Dprimi_{N0}$ one must consider, in addition to the numbers M_{N0} less than N_0 and incongruous with it, also the pi belonging to the set $\mathbb{P}\left(\sqrt{(N_0)}\right)$ and consequently *Dprimi* always results$_{N0} > Dnc_{N0}$. Let us then calculate the error that is made by setting Dprimi = Dnc_{N0} . According to NPT (2.1) we can write:

$$(2.2.6)\quad Dnc_{]0,N_0]} = \frac{\left(\frac{N_0}{logN_0} - \frac{\sqrt{N_0}}{log\sqrt{N_0}}\right)}{N_0} \qquad \text{e } Dprimi_{]0,N_0]} = \frac{1}{logN_0}$$

Observation 2.2.6 *Having ascertained that it always results $Dprimi_{]0,N_0]} > Dnc_{]0,N_0]}$ one can easily calculate that the percentage error in positing $Dprimi_{]0,N_0]} = Dnc_{]0,N_0]}$ is 20% for $N_0 = 10^2$, 2% for $N_0 = 10^4$, 0.02% for $N_0 = 10^8$, and that it is gradually decreasing for increasing values of N_0 .*

APPENDIX A

Table showing the bi-univocal correspondence between the numbers 1 to 210 and all combinations of the congruence classes modulo 2 - 3 - 5- 7		
no. modules	**no. modules**	**no. modules**
2 - 3 - 5 - 7	2 - 3 - 5 - 7	2 - 3 - 5 - 7
1) 1 - 1 - 1 - 1	71) 1 - 2 - 1 - 1	141) 1 - 0 - 1 - 1
2) 0 - 2 - 2 - 2	72) 0 - 0 - 2 - 2	142) 0 - 1 - 2 - 2
3) 1 - 0 - 3 - 3	73) 1 - 1 - 3 - 3	143) 1 - 2 - 3 - 3
4) 0 - 1 - 4 - 4	74) 0 - 2 - 4 - 4	144) 0 - 0 - 4 - 4
5) 1 - 2 - 0 - 5	75) 1 - 0 - 0 - 5	145) 1 - 1 - 0 - 5
6) 0 - 0 - 1 - 6	76) 0 - 1 - 1 - 6	146) 0 - 2 - 1 - 6
7) 1 - 1 - 2 - 0	77) 1 - 2 - 2 - 0	147) 1 - 0 - 2 - 0
8) 0 - 2 - 3 - 1	78) 0 - 0 - 3 - 1	148) 0 - 1 - 3 - 1
9) 1 - 0 - 4 - 2	79) 1 - 1 - 4 - 2	149) 1 - 2 - 4 - 2
10) 0 - 1 - 0 - 3	80) 0 - 2 - 0 - 3	150) 0 - 0 - 0 - 3
11) 1 - 2 - 1 - 4	81) 1 - 0 - 1 - 4	151) 1 - 1 - 1 - 4
12) 0 - 0 - 2 - 5	82) 0 - 1 - 2 - 5	152) 0 - 2 - 2 - 5
13) 1 - 1 - 3 - 6	83) 1 - 2 - 3 - 6	153) 1 - 0 - 3 - 6
14) 0 - 2 - 4 - 0	84) 0 - 0 - 4 - 0	154) 0 - 1 - 4 - 0
15) 1 - 0 - 0 - 1	85) 1 - 1 - 0 - 1	155) 1 - 2 - 0 - 1
16) 0 - 1 - 1 - 2	86) 0 - 2 - 1 - 2	156) 0 - 0 - 1 - 2
17) 1 - 2 - 2 - 3	87) 1 - 0 - 2 - 3	157) 1 - 1 - 2 - 3
18) 0 - 0 - 3 - 4	88) 0 - 1 - 3 - 4	158) 0 - 2 - 3 - 4
19) 1 - 1 - 4 - 5	89) 1 - 2 - 4 - 5	159) 1 - 0 - 4 - 5
20) 0 - 2 - 0 - 6	90) 0 - 0 - 0 - 6	160) 0 - 1 - 0 - 6
21) 1 - 0 - 1 - 0	91) 1 - 1 - 1 - 0	161) 1 - 2 - 1 - 0
22) 0 - 1 - 2 - 1	92) 0 - 2 - 2 - 1	162) 0 - 0 - 2 - 1
23) 1 - 2 - 3 - 2	93) 1 - 0 - 3 - 2	163) 1 - 1 - 3 - 2
24) 0 - 0 - 4 - 3	94) 0 - 1 - 4 - 3	164) 0 - 2 - 4 - 3
25) 1 - 1 - 0 - 4	95) 1 - 2 - 0 - 4	165) 1 - 0 - 0 - 4
26) 0 - 2 - 1 - 5	96) 0 - 0 - 1 - 5	166) 0 - 1 - 1 - 5
27) 1 - 0 - 2 - 6	97) 1 - 1 - 2 - 6	167) 1 - 2 - 2 - 6
28) 0 - 1 - 3 - 0	98) 0 - 2 - 3 - 0	168) 0 - 0 - 3 - 0
29) 1 - 2 - 4 - 1	99) 1 - 0 - 4 - 1	169) 1 - 1 - 4 - 1
30) 0 - 0 - 0 - 2	100) 0 - 1 - 0 - 2	170) 0 - 2 - 0 - 2
31) 1 - 1 - 1 - 3	101) 1 - 2 - 1 - 3	171) 1 - 0 - 1 - 3
32) 0 - 2 - 2 - 4	102) 0 - 0 - 2 - 4	172) 0 - 1 - 2 - 4
33) 1 - 0 - 3 - 5	103) 1 - 1 - 3 - 5	173) 1 - 2 - 3 - 5
34) 0 - 1 - 4 - 6	104) 0 - 2 - 4 - 6	174) 0 - 0 - 4 - 6
35) 1 - 2 - 0 - 0	105) 1 - 0 - 0 - 0	175) 1 - 1 - 0 - 0
36) 0 - 0 - 1 - 1	106) 0 - 1 - 1 - 1	176) 0 - 2 - 1 - 1
37) 1 - 1 - 2 - 2	107) 1 - 2 - 2 - 2	177) 1 - 0 - 2 - 2
38) 0 - 2 - 3 - 3	108) 0 - 0 - 3 - 3	178) 0 - 1 - 3 - 3
39) 1 - 0 - 4 - 4	109) 1 - 1 - 4 - 4	179) 1 - 2 - 4 - 4
40) 0 - 1 - 0 - 5	110) 0 - 2 - 0 - 5	180) 0 - 0 - 0 - 5
41) 1 - 2 - 1 - 6	111) 1 - 0 - 1 - 6	181) 1 - 1 - 1 - 6
42) 0 - 0 - 2 - 0	112) 0 - 1 - 2 - 0	182) 0 - 2 - 2 - 0
43) 1 - 1 - 3 - 1	113) 1 - 2 - 3 - 1	183) 1 - 0 - 3 - 1
44) 0 - 2 - 4 - 2	114) 0 - 0 - 4 - 2	184) 0 - 1 - 4 - 2
45) 1 - 0 - 0 - 3	115) 1 - 1 - 0 - 3	185) 1 - 2 - 0 - 3
46) 0 - 1 - 1 - 4	116) 0 - 2 - 1 - 4	186) 0 - 0 - 1 - 4
47) 1 - 2 - 2 - 5	117) 1 - 0 - 2 - 5	187) 1 - 1 - 2 - 5
48) 0 - 0 - 3 - 6	118) 0 - 1 - 3 - 6	188) 0 - 2 - 3 - 6
49) 1 - 1 - 4 - 0	119) 1 - 2 - 4 - 0	189) 1 - 0 - 4 - 0
50) 0 - 2 - 0 - 1	120) 0 - 0 - 0 - 1	190) 0 - 1 - 0 - 1
51) 1 - 0 - 1 - 2	121) 1 - 1 - 1 - 2	191) 1 - 2 - 1 - 2
52) 0 - 1 - 2 - 3	122) 0 - 2 - 2 - 3	192) 0 - 0 - 2 - 3
53) 1 - 2 - 3 - 4	123) 1 - 0 - 3 - 4	193) 1 - 1 - 3 - 4
54) 0 - 0 - 4 - 5	124) 0 - 1 - 4 - 5	194) 0 - 2 - 4 - 5
55) 1 - 1 - 0 - 6	125) 1 - 2 - 0 - 6	195) 1 - 0 - 0 - 6
56) 0 - 2 - 1 - 0	126) 0 - 0 - 1 - 0	196) 0 - 1 - 1 - 0
57) 1 - 0 - 2 - 1	127) 1 - 1 - 2 - 1	197) 1 - 2 - 2 - 1
58) 0 - 1 - 3 - 2	128) 0 - 2 - 3 - 2	198) 0 - 0 - 3 - 2
59) 1 - 2 - 4 - 3	129) 1 - 0 - 4 - 3	199) 1 - 1 - 4 - 3
60) 0 - 0 - 0 - 4	130) 0 - 1 - 0 - 4	200) 0 - 2 - 0 - 4
61) 1 - 1 - 1 - 5	131) 1 - 2 - 1 - 5	201) 1 - 0 - 1 - 5
62) 0 - 2 - 2 - 6	132) 0 - 0 - 2 - 6	202) 0 - 1 - 2 - 6
63) 1 - 0 - 3 - 0	133) 1 - 1 - 3 - 0	203) 1 - 2 - 3 - 0
64) 0 - 1 - 4 - 1	134) 0 - 2 - 4 - 1	204) 0 - 0 - 4 - 1
65) 1 - 2 - 0 - 2	135) 1 - 0 - 0 - 2	205) 1 - 1 - 0 - 2
66) 0 - 0 - 1 - 3	136) 0 - 1 - 1 - 3	206) 0 - 2 - 1 - 3
67) 1 - 1 - 2 - 4	137) 1 - 2 - 2 - 4	207) 1 - 0 - 2 - 4
68) 0 - 2 - 3 - 5	138) 0 - 0 - 3 - 5	208) 0 - 1 - 3 - 5
69) 1 - 0 - 4 - 6	139) 1 - 1 - 4 - 6	209) 1 - 2 - 4 - 6
70) 0 - 1 - 0 - 0	140) 0 - 2 - 0 - 0	210) 0 - 0 - 0 - 0

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