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Investigation of Hardy-Littlewood conjecture with the primality theorems of Congruence and of Complementary Congruence and with the density of incongruous and incompcongruous numbers

Abstract

A study of the Hardy-Littlewood conjecture (infinity and distribution of prime twins) is developed in the article; it is based primarily on the two primality theorems of congruence and compcongruence. The study leads to the proof of the Hardy-Littlewood conjecture and, in addition to the results achieved, opens up new areas of possible research in the field of Number Theory.

1 The Twin Primes and the Hardy-Littlewood conjecture

As is well known, twin primes are those that are 2 units apart (except for the pair 2-3)
such as 17-19, 71-73, 521-523, 6359-6361, etc.

The Hardy-Littlewood conjecture states that first twins are infinite.

1.1 The Twin Numbers

Every prime number greater than 2, as well as every odd number, can be written as the sum or difference of an even number and 1. In the case of a pair of prime twins there will obviously be a single even number which when added to 1 and subtracted by 1 will give rise to the prime twins of the pair.

We call an even twin and denote by the symbol PG every even number n ∈ N such that n+1 and n-1 are prime numbers.

1.2 The Equal Twins Theorem

Definition 1.2.1 *$\forall n_0 \in N$, even and greater than 4, with $\mathbb{P}\left(\sqrt{(n_0+1)}\right)$ set of odd prime numbers $\leq\sqrt{(n_0+1)}$, a necessary and sufficient condition for n_0 +1 and n_0 -1 to be twin primes is that $n_0 \not\equiv 1 (mod\, p_i)$ and $n_0 \not\Vdash 1$[1] $(mod\, p_i)$ $\forall p_i \in \mathbb{P}\left(\sqrt{(n_0+1)}\right)$ or that $\mathbb{P}\left(\sqrt{(n_0+1)}\right)$ is an empty set.*

DIM. From the two Primality Theorems (1.2.1 and 1.4.1 [c]) positing $N_0 = n_0$ and $n_0 = 1$ it follows that, if 1 is incongruous and incompcongruous with n_0 p modules$_i$ $\forall p_i \in \mathbb{P}\left(\sqrt{(n_0+1)}\right)$, e di conseguenza $\forall p_i \in \mathbb{P}\left(\sqrt{(n_0-1)}\right)$ essendo $\mathbb{P}\left(\sqrt{(n_0-1)}\right) \subseteq \mathbb{P}\left(\sqrt{(n_0+1)}\right)$, or if $\mathbb{P}\left(\sqrt{(n_0+1)}\right)$ is an empty set, n_0 +1 and n_0 - 1 are twin primes.
Conversely, if n_0 + 1 and n_0 - 1 are twin primes, this means that they are not divisible by any prime less than or equal to the $\sqrt{(n_0+1)}$ and that therefore, again by (1.2.1) and (1.4.1), n_0 and 1 are incongruous and incompcongruous $\forall p_i \in \mathbb{P}\left(\sqrt{(n_0+1)}\right)$ and therefore $\forall p_i \in \mathbb{P}\left(\sqrt{(n_0-1)}\right)$.
We set $n_0 \geq 4$ because with $n_0 = 2$ we would have that $n_0 - 1 = 1$ which, as we know, is neither a prime nor a compound number.

[1] The symbols ≢ and ⫳ stand for non-congruity and non-compcongruity between two numbers from: Congruity, Primality and Density by Aldo Pappalepore

If instead of referring to the set $\mathbb{P}\left(\sqrt{(n_0+1)}\right)$ we refer, for the sake of subsequent demonstrations, to the set $\mathbb{P}\left(\sqrt{N_0}\right)$ with $N_0 \in N$ e maggiore di $n_0$the theorem (3.2.1) is transformed into the following corollary:

Corollary 1.2.2 *$\forall N_0$, $n_0 \in N$, with $N_0 \geq 9$ and with n_0 even and $p_{max} < n_0 < N_0$, with $\mathbb{P}\left(\sqrt{N_0}\right)$ set of odd prime numbers $\leq \sqrt{(N_0)}$ and with p_{max} higher prime number than $\mathbb{P}\left(\sqrt{N_0}\right)$, a necessary and sufficient condition for n_0 +1 and n_0 -1 to be twin primes is that $\forall p_i \in \mathbb{P}\left(\sqrt{N_0}\right)$ 1 is an incongruous and incompcongruous number of n_0 .*

Dim. substituting $\mathbb{P}\left(\sqrt{(N_0)}\right)$ a $\mathbb{P}\left(\sqrt{(n_0+1)}\right)$ the numbers n_0 even less than p_{max} and such that $n_0 \pm 1 = p_j$, with $p_j \in \mathbb{P}\left(\sqrt{(N_0)}\right)$, are not taken into account since, for the same $p_j \in \mathbb{P}\left(\sqrt{(N_0)}\right)$, have a class of congruence mod p_j equal and/or complementary to that of equal modulus of 1. In fact, if $n_0 \pm 1 = p_j$ according to modular arithmetic it will always be the case that $[n_0]$ mod $p_j \pm [1]$ mod $p_j = [p_j]$ mod $p_j = [0]$ from which the congruence and/or compcongruence mod p_j of 1 with n_0 follows.
Conversely, if n_0 + 1 and n_0 - 1 are twin primes greater than p_{max} and less than N_0 it means both that, according to (1.2.1) and (1.4.1), n_0 and 1 are incongruous and incompcongruous $\forall p_i \in \mathbb{P}\left(\sqrt{(n_0 \pm 1)}\right)$, but also that, since n_0 + 1 and n_0 - 1, as primes, are not divisible by any prime less than or equal to $\sqrt{(N_0)}$, n_0 and 1 are incongruous and incompcongruous anche $\forall p_i \in \mathbb{P}\left(\sqrt{(N_0)}\right)$.
He placed himself $N_0 \geq 9$ in quanto per valori inferiori p_{max} would not be defined.

Since in the interval $]0, N_0]$, with $N_0 \geq 9$ and $n_0 > p_{max}$ there is always at least one prime (observation 1.2.5 [c]), surely there will always exist an n_{01} and an n_{02} of which 1 is incongruous and incompcongruous; but in order to prove the Hardy-Littlewood conjecture, one must ascertain both that there exists at least one $n = n_0 = _{01} \ _{02}$, ni.e. an even twin number (PG), smaller than N_0 , of which 1 is incongruous and incompcongruous modulo p_i for all p_i belonging to the ***set*** $\mathbb{P}\left(\sqrt{(\boldsymbol{N_0})}\right)$, let it be that for $N_0 \rightarrow \infty$ also the number of PGs tends to infinity with a definite relation.
To this end, we resort to the study of the density of twin peers.

1.3 The density of twin peers

All n_0 that satisfy the conditions of corollary (1.2.2) are even PG twins with the following characteristics:

- the class of PG module 2, PG being even, is always zero while the class of 1 module 2 is always 1 (with complement equal to 1) and consequently 1 will always be incongruous and incompcongruous with PG module 2
- PG classes of the next module (3, 5, 7, 11, etc.) present in $\mathbb{P}\left(\sqrt{(N_0)}\right)$ must not be equal to the classes of 1 and their p-1 complements of the same module (*e.g. if PG=18 and $N_0 = 24$ we have that $\mathbb{P}(\sqrt{N_0}) = \{3\}$; $[18]_{mod3} = 0$ and its complement is still equal to 0, $[1]_{mod3} = [1]$ and its complement is equal to 2 and therefore 1 is incongruous and incompcongruous with PG so that 18+1 and 18-1 are twin primes*).

Having said this, let us see how to calculate the number of PGs and thus pairs of even twins less than a $N_0 \geq 49$, a condition (see observation 1.7.1 [c]) resulting from the necessity that N_0 belongs to the interval $]0, p_{max}\#]$ where p_{max} is the highest prime number less than or equal to the $\sqrt{(N_0)}$.
Having then selected any $N_0 \geq 49$ we denote by p_{max} the highest prime number of $\mathbb{P}\left(\sqrt{(N_0)}\right)$. Let us then consider the interval/table of natural numbers $]0, p_{max}\#]$ and now eliminate from this table the

rows that have: congruence class mod 2 equal to [1]; congruence classes of successive modules (3, 5, ... , p_{max}) equal to the classes of 1 and their complements p-1 for the same modules.
The numbers **M of** the number-class table p_{max} , not eliminated through the previous sieve, can then only be those which in their corresponding combination of congruence classes present only the class [0] of the two possible congruence classes mod 2 and one of the p_i -2 (for each p_i belonging to the set $\mathbb{P}(\sqrt{(N_0)})$) possible congruence classes of the successive modules (3, 5, ... , p_{max}) with the exclusion, that is, of classes 1 and (p-1) for the same modules (if e.g. (M) mod7 = 1 with complement = 6, M will not be an Equal Twin being also (1) mod7=1 with complement = 6; to be so, it is necessary that (M) mod7 is equal to one of the 5 (7-2) other possible congruence classes: 0,2,3,4,5)

The rows (combinations of classes) of the table that have not been deleted will then, according to combinatorial calculation, be:

(1.3.1) $\prod_{p=3}^{p_{max}}(p-2)$

Therefore (1.3.1) gives us the quantity of the numbers **M of** the table-interval]0, p_{max} #] of which 1 **is not congruent and is not compcongruent only for the modules p_i** belonging to the set $\mathbb{P}(\sqrt{(N_0)})$ while nothing can be said about the possible (non) congruence and (non) compcongruence of 1 with these numbers with respect to the other modules p_j greater than p_{max} and belonging to the set $\mathbb{P}(\sqrt{(pmax\#)})$). According to the corollary (1.2.2), however, we can state that all numbers **M less than N_0** , which we will denote by **PG_{N0}** , being non-congruent and non-compcongruent with 1 for all modules p_i belonging to the set $\mathbb{P}(\sqrt{(N_0)})$, are Even Twin numbers. We denote by $Dncncomp(PG)_{N0}$ their density in the interval]0, N].0

Remark 1.3.2 *By the same corollary (1.2.2) we also know, however, that these numbers M(PG), of which 1 is not compcongruent with respect to the modules* $\mathbb{P}(\sqrt{(N_0)})$, *do not include the PGs relative to the pairs of primes less than the* $\sqrt{N_0}$ *and consequently their average density* $Dncncomp(PG)_{N0}$ *will always be lower than that* Dpg_{N0} *of all pairs of P twins*G_{N0} *less than N_0 .*

Let us now calculate the density of PG numbers existing in the interval]0, p_{max} #] of which 1 is non-congruent and non-compcongruent for the p-modules$_i$ belonging to the set $\mathbb{P}(\sqrt{(N_0)})$, and we denote this density by $Dncncomp_{]0,\ p_{max}\#]}$ or with $Dncncomp_{]0,\ \sqrt{(N_0)}\#]}$ being $\sqrt{(N_0)}\# = p_{max}\#$].
Knowing then that p_{max} # = 2*3*.......* p_{max} , we can write:

(1.3.3) $$Dncncomp_{]0,\ \sqrt{(N_0)}\#]} = \frac{\prod_{p=3}^{pmax}(p-2)}{\prod_{p=2}^{pmax} p} = \frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-2)}{p}$$

By multiplying and dividing the second term of the same by (p-1) we obtain:

(1.3.4) $$Dncncomp_{]0,\ \sqrt{(N_0)}\#]} = \frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-2)}{p} * \frac{(p-1)}{(p-1)} = \frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-1)}{p} * \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} = \prod_{p=2}^{pmax}\frac{(p-1)}{p} * \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)}$$

In the last member of (1.3.4) we have substituted for $\frac{1}{2} * \prod_{p=3}^{pmax}\frac{(p-1)}{p}$ the term $\prod_{p=2}^{pmax}\frac{(p-1)}{p}$ which, as we know from (2.2.2 [c]), corresponds, always for $N_0 \geq 49$, to the average density $Dnc_{]0,\ \sqrt{N_0}\#]}$ of the numbers M existing in the interval]0, p_{max} #] **not congruent with N_0 for only p_i** belonging to the set $\mathbb{P}(\sqrt{(N_0)})$; in these last three formulae p_{max} is the highest prime number less than or equal to the $\sqrt{(N_0)}$.

Let us then see if we can find a relationship between $\prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)}$ e $\prod_{p=2}^{pmax}\frac{(p-1)}{p}$ so that we can determine the value of $Dncncomp_{]0,\ p_{max}\,\#]}$ as a function of $Dnc_{]0,\ \sqrt{N}_0\,\#]}$.

We can write:

(1.3.5) $$\frac{\prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)}}{\prod_{p=2}^{pmax}\frac{(p-1)}{p}} = \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} * \prod_{p=2}^{pmax}\frac{p}{(p-1)} = \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} * 2 * \prod_{p=3}^{pmax}\frac{p}{(p-1)} = 2 * \prod_{p=3}^{pmax}\frac{p*(p-2)}{(p-1)^2}$$

and substituting (1.3.5) into (1.3.4):

(1.3.6) $$Dncncomp_{]0,\ \sqrt{(N_0)}\,\#]} = \prod_{p=2}^{pmax}\frac{(p-1)}{p} * \prod_{p=3}^{pmax}\frac{(p-2)}{(p-1)} = 2 * \prod_{p=3}^{pmax}\frac{p*(p-2)}{(p-1)^2} * \left(\prod_{p=2}^{pmax}\frac{(p-1)}{p}\right)^2$$

from which according to (2.2.2 [c]):

(1.3.7) $$Dncncomp_{]0,\ \sqrt{(N_0)}\,\#]]} \approx 2 * \prod_{p=3}^{pmax}\frac{p*(p-2)}{(p-1)^2} * (Dnc_{]0,\ \sqrt{N}_0\,\#]})^2$$

From (1.3.7) we derive the ratio between the density $Dncncomp_{]0,\ \sqrt{(N_0)}\,\#]}$ of the incongruous and incompcongruous numbers with 1 in the interval]0. $\sqrt{N_0}$ #] and the square of that in the same interval of the incongruous numbers with N_0.

(1.3.8) $$\frac{Dncncomp_{]0,\ \sqrt{(N_0)}\,\#]}}{(Dnc_{]0,\ \sqrt{N}_0\,\#]})^2} \approx 2 * \prod_{p=3}^{pmax}\frac{p*(p-2)}{(p-1)^2}$$

Now it can easily be verified that the term $\prod_{p=3}^{pmax}\frac{p*(p-2)}{(p-1)^2}$ for N_0 =49 assumes the value 0.68359375, for N_0 =9006001 the value 0.6601862196 and then, as N increases$_0$ towards infinity, and thus extending the product over all prime numbers ≥ 3, it tends to decrease rapidly towards the constant of the prime twins C_2 that appears in the Hardy-Littlewood conjecture ([d]) on the distribution of prime twins:

$$\prod_{p\geq 3}\frac{p*(p-2)}{(p-1)^2} = C_2 \approx 0.66016118138468695739278121100 14 ....$$

We can therefore write:

(1.3.9) $$\frac{Dncncomp_{]0,\ \sqrt{(N_0)}\,\#]}}{(Dnc_{]0,\ \sqrt{N}_0\,\#]})^2} \approx 2*C_2$$

Giacché:

- the ratio $2C_2$ between $Dncncomp_{]0,\ \sqrt{(N_0)}\,\#]}$ and the square of $Dnc_{]0,\ \sqrt{N}_0\,\#]}$ changes little as N varies$_0$ and thus of $\sqrt{N_0}$ #
- the relationship between the prime twins and the prime numbers depends only on the distribution of the latter, i.e. on $\prod_{p=2}^{p_{max}}\frac{(p-1)}{p} = Dnc_{]0,\ \sqrt{N_0}\,\#]}$

- the sieve determining the numbers n_0 with which 1 is incongruous and incompcongruous depends neither on N_0 nor on $\sqrt{N_0}\#$ but only on the incongruity and incompcongruity of 1 with such n_0 $\forall p_i \in \mathbb{P}(\sqrt{(n_0 \pm 1)})$

it may be assumed with good approximation that the relation (1.3.9) is valid for any interval]0, N] and thus in particular also for the interval]0, N_0] and that it is therefore correct to write:

(1.3. 10) $Dncncomp_{]0,\ N_0]\ 2} * 2*C \approx (Dnc_{]0,N_0]})^2$

Comment 1.3.11 *In (1.3.10) as stated in Comment (1.3.2) and (2.2.6 [c]) both* $Dncncomp_{]0,\ N_0]}$ *and* $Dnc_{]0,N_0]}$ *do not include the possible* n_0 *for which* $n_0 \pm 1$ *are equal to the primes less than or equal to the* $\sqrt{(N_0)}$ *but since this relation is always valid* $\forall N_0 \in N$ *starting from* $N_0 = 49$ *we can extend (*1.3.10*) to all the numbers* n_0 *of which 1 is not congruous and not compcongruous and which when added to or subtracted from 1 result in the primes (except 2, 3, 5, 7) less than any* N_0 *greater than 49. In fact for* $N_0 = 49$ *(and therefore* $\sqrt{49} = 7$*) the (*1.3.10*)* concerns all the n_0 of which 1 is *not congruous and not compcongruous* that subtracted and added to 1 give as result the first twins *between 8 and 49; for* $N_0 = 121$ *(and therefore* $\sqrt{121} = 11$*) the (*1.3.10*) concerns the first twins between 12 and 121; for* $N_0 = 169$ *(and thus* $\sqrt{169} = 13$*), (*1.3.10*) concerns the first twins between 14 and 169; and we can continue in this manner for all subsequent* N_0 *equal to the squares of the first twins after 13.*

But it can be verified, assuming N_0 =49 and thus C_2 =0.6835, that (1.3.10) with an approximation of about 5%, also subsists by taking into account primes 2, 3, 5, 7. In fact, with N_0 =49 there are 15 primes and 6 pairs of twins, hence, indicating with $\mathrm{Dpg}_{\mathrm{N0}}$ the density of even twins less than 49, we have:

$$\mathrm{Dpg}_{\mathrm{N0}} = \frac{6}{49} = 0{,}1224$$

while for (1.3.10):

$$Dncncomp_{]0,\ N_0]} \approx 2*0{,}6835*\left(\frac{15}{49}\right)^2 \approx 0{,}1281$$

whence:

$$Dncncomp_{]0,\ N_0]} \approx \mathrm{Dpg}_{\mathrm{N0}}$$

Obviously, as N increases$_0$, (1.3.10) being valid for all primes greater than 7, the approximation decreases.

Ultimately, we can then hold that $\forall N_0 \in N$ greater than 49, (1.3.10) is valid for all primes less than N_0 and therefore, substituting $\mathrm{Dpg}_{\mathrm{N0}}$ al posto di $Dncncomp_{]0,\ N_0]}$ e $Dprimi_{]0,N_0]}$ al posto di $Dnc_{]0,N_0]}$ (see (2.2.4) [c]), writing:

(1.3.11) $\mathrm{Dpg}_{\mathrm{N0}} \approx 2*C_2*(Dprimi_{]0,N_0]})^2$

Being then for the NPT $Dprimi_{]0,N_0]} = \frac{1}{logN_0}$ one can write:

(1.3.12) $\mathrm{Dpg}_{\mathrm{N0}} \approx 2*C_2*(\frac{1}{logN_0})^2$

and multiplying both members by N_0 :

(1.3.13) $PG_{N0} \approx N *2*C *_{02}(\frac{1}{logN_0})^2$

For N_0 = 49 the (1.3.13) PG_{N0} takes a value greater than 5 and, since $N *_0(\frac{1}{logN_0})^2$ an increasing function with N_0 , PG_{N0} will always grow as N_0 tends to infinity with a distribution **(** 1.3.12) equal to that predicted by the Hardy-Littlewood conjecture [(d)]:

$\pi_2 (x) \approx x*2*C *_2(\frac{1}{logx})^2$

The even twins, i.e. pairs of prime twins, are therefore infinite and (1.3.12) is their distribution law.

BIBLIOGRAPHY

[a] Alessandro Zaccagnini - Introduction to Analytical Number Theory:
http://people.dmi.unipr.it/alessandro.zaccagnini/psfiles/lezioni/tdn2005.pdf

[b] Francesco Fumagalli - Appunti di Teoria elementare dei numeri:
Theory_of_Numbers.pdf (unifi.it)

[c] Aldo Pappalepore - Congruence, Primality and Density:
https://www.aldopappalepore.it/_downloads/99fd430c6fe12b90fb801cd67dc1d70f

[d] HARDY-LITTLEWORD - The conjecture of the first twins:
Wikizero - Congruction of twin primes